ON THE RELATIONSHIP OF AP, RS AND CEP IN CONGRUENCE MODULAR VARIETIES. II

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Abstract. Let $\mathcal{V}$ be a congruence distributive variety, or a congruence modular variety whose free algebra on 2 generators is finite. If $\mathcal{V}$ is residually small and has the amalgamation property, then it has the congruence extension property. Several applications are presented.

In two previous papers [1] and [2], we considered the following question: if $\mathcal{V}$ is a residually small variety with the amalgamation property, must $\mathcal{V}$ have the congruence extension property? Our work established the following implications for a congruence modular variety $\mathcal{V}$:

(1) If $\mathcal{V}$ is 2–finite and has C2, then $\mathrm{AP} + \mathrm{RS} \implies \mathrm{R}$.
(2) If $\mathcal{V}$ is 4–finite with C2 and R, then $\mathrm{AP} + \mathrm{RS} \implies \mathrm{CEP}$.

(The terminology will be explained below.)

In this paper we supplement and extend these results. Assuming still that $\mathcal{V}$ is congruence modular, we have:

(3) $\mathrm{AP} + \mathrm{RS} \implies \mathrm{C2}$.
(4) If $\mathcal{V}$ has R, then $\mathrm{AP} + \mathrm{RS} \implies \mathrm{CEP}$.

Combining these implications, we have that every congruence modular, 2–finite variety satisfies $\mathrm{AP} + \mathrm{RS} \implies \mathrm{CEP}$. Furthermore, every congruence distributive variety (no finiteness assumption) satisfies $\mathrm{AP} + \mathrm{RS} \implies \mathrm{CEP}$.

Our universal algebraic notation and terminology are standard. Good references are [4] and [9].

Let $\mathcal{V}$ be a variety of algebras. We say that $\mathcal{V}$

- has the *amalgamation property (AP)* if, for all $\mathbf{A}, \mathbf{B}_0, \mathbf{B}_1 \in \mathcal{V}$ and all embeddings $f_i : \mathbf{A} \to \mathbf{B}_i$, for $i = 0, 1$, there is $\mathbf{C} \in \mathcal{V}$ and embeddings $g_i : \mathbf{B}_i \to \mathbf{C}$, $i = 0, 1$, such that $g_0 \circ f_0 = g_1 \circ f_1$,
- is *residually small (RS)* if there is a cardinal κ such that every subdirectly irreducible algebra in $\mathcal{V}$ has cardinality less than κ,
- has the *congruence extension property (CEP)* if, for all $\mathbf{A} \leq \mathbf{B} \in \mathcal{V}$, and congruence α on $\mathbf{A}$, there is $\bar{\alpha} \in \operatorname{Con} \mathbf{B}$ such that $\bar{\alpha} \restriction \mathbf{A} = \alpha$,
- is *n–finite*, for a positive integer n, if every member of $\mathcal{V}$ generated by n elements is finite.

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The least non-trivial congruence on a subdirectly irreducible algebra is called the *monolith*, and the least and greatest congruences on an algebra are denoted 0 and 1 respectively. Let $\mathbf{B}$ be a subalgebra of $\mathbf{A}_0 \times \mathbf{A}_1$. It is often convenient to denote the elements of $\mathbf{B}$ as vertical pairs $\binom{x_0}{x_1}$ with $x_i \in A_i$ for $i = 0, 1$. Let $\alpha \in \operatorname{Con} \mathbf{A}_0$. Then there is an induced congruence α_0 on $\mathbf{B}$ given by $\binom{x_0}{x_1} \alpha_0 \binom{y_0}{y_1} \iff x_0 \ \alpha \ y_0$. Similarly, for $\beta \in \operatorname{Con} \mathbf{A}_1$, there is a congruence β_1 on $\mathbf{B}$ defined in an analogous way. The particularly important congruences 0_0 and 0_1 are denoted η_0 and η_1 respectively. Given a congruence β on an algebra $\mathbf{D}$, we write $\mathbf{D}(\beta)$ for the subalgebra of $\mathbf{D}^2$ with universe β. That is: $D(\beta) = \left\{ \binom{x}{y} : x \ \beta \ y \right\}$.

The commutator is a binary operation, denoted $[\ \ ,\ \ ]$, defined on the congruence lattice of an algebra. In a congruence modular variety $\mathcal{V}$, the commutator has some strong properties. It is *additive*, that is

$$[\,\alpha, \bigvee \Theta\,] = \bigvee_{\theta \in \Theta} [\,\alpha, \theta\,] \quad \text{for } \mathbf{A} \in \mathcal{V},\ \alpha \in \operatorname{Con} \mathbf{A}, \text{ and } \Theta \subseteq \operatorname{Con} \mathbf{A},$$

symmetric and *meet-dominated*

$$[\,\alpha, \beta\,] = [\,\beta, \alpha\,] \subseteq \alpha \wedge \beta \quad \text{for } \mathbf{A} \in \mathcal{V} \text{ and } \alpha, \beta \in \operatorname{Con} \mathbf{A},$$

respects finite direct products

$$\begin{aligned} &\text{for } \alpha, \beta \in \operatorname{Con} \mathbf{A} \text{ and } \gamma, \delta \in \operatorname{Con} \mathbf{B} \\ &\qquad [\,\alpha_0 \wedge \gamma_1, \beta_0 \wedge \delta_1\,] = [\,\alpha, \beta\,]_0 \wedge [\,\gamma, \delta\,]_1 \text{ in } \mathbf{A} \times \mathbf{B}, \end{aligned}$$

and exhibits the following *homomorphism property*

$$\begin{aligned} &\text{if } f : \mathbf{A} \to \mathbf{B} \text{ is a surjective homomorphism and } \alpha, \beta \in \operatorname{Con} \mathbf{B} \text{ then} \\ &\qquad f^{-1} [\,\alpha, \beta\,] = \left[\,f^{-1}\alpha, f^{-1}\beta\,\right] \vee \ker f. \end{aligned}$$

For a systematic development of the subject, we direct the reader to [6], especially chapter 4.

There are three identities involving the commutator that we consider in this paper. We say that an algebra $\mathbf{A}$ has:

$$\begin{array}{lll} \text{C1 if} & \forall \alpha, \beta \in \operatorname{Con} \mathbf{A} & \alpha \wedge [\,\beta, \beta\,] = [\,\alpha \wedge \beta, \beta\,] \\ \text{C2 if} & \forall \alpha, \beta \in \operatorname{Con} \mathbf{A} & [\,\alpha, \beta\,] = \alpha \wedge \beta \wedge [\,1, 1\,] \\ \text{R if} & \forall \mathbf{B} \leq \mathbf{A} & [\,1_{\mathbf{A}}, 1_{\mathbf{A}}\,] \restriction \mathbf{B} = [\,1_{\mathbf{B}}, 1_{\mathbf{B}}\,]. \end{array}$$

We say that a class $\mathcal{K}$ has one of these properties if and only if every member of $\mathcal{K}$ has the property. Finally for congruences α, β on an algebra $\mathbf{A}$, we define $(\alpha : \beta)$ to be the largest congruence θ such that $[\,\beta, \theta\,] \leq \alpha$.

We collect some facts connecting these concepts in the following theorem.

Theorem 0. *Let $\mathcal{V}$ be a congruence modular variety.*

(1) *$\mathcal{V}$ satisfies C1 if and only if, for every subdirectly irreducible algebra $\mathbf{A}$ of $\mathcal{V}$ with monolith β, if $\theta = (0 : \beta)$ then $[\,\theta,\, \theta\,] = 0$.*
(2) *$\mathcal{V}$ satisfies C2 if and only if every subdirectly irreducible algebra $\mathbf{A}$ of $\mathcal{V}$ satisfies C2, and this in turn is equivalent to $\mathbf{A}$ being either abelian or prime.*
(3) *C2 $\implies$ C1.*
(4) *If $\mathcal{V}$ is a residually small variety then it satisfies C1. Conversely, if $\mathcal{V}$ is finitely generated and satisfies C1, then it is residually small.*
(5) *Every variety with CEP satisfies C2 + R.*

(1) and (4) are from [5], (2) and (5) are from [7]. ($\mathbf{A}$ is *prime* iff $\alpha, \beta \neq 0 \implies [\,\alpha,\, \beta\,] \neq 0$). (3) follows easily from the definitions.

Our first objective is the following

Theorem 1. *Let $\mathcal{V}$ be a congruence modular variety. If $\mathcal{V}$ has AP and RS, then $\mathcal{V}$ has C2.*

We require a pair of lemmas. The first of these is virtually identical to Theorem 10.9 of [6]. We include the proof for completeness.

Lemma 2. *Let $\mathcal{V}$ be a congruence modular, residually small variety. If $\mathcal{V}$ fails C2, then $\mathcal{V}$ contains a subdirectly irreducible algebra $\mathbf{A}$ with monolith β, and an endomorphism f such that*

(1) $0 = [\,\beta,\, \beta\,] < \beta = [\,\beta,\, 1\,]$
(2) $f = f^2$
(3) $x \mathrel{\beta} y \iff f(x) = f(y) \mathrel{\beta} y$.

Proof. By Theorem 0, $\mathcal{V}$ satisfies C1 since it is residually small. If $\mathcal{V}$ fails C2, there is a subdirectly irreducible algebra $\mathbf{D}$ which is neither abelian nor prime. Let γ be the monolith of $\mathbf{D}$ and $\kappa = (0 : \gamma)$. Then, since $\mathbf{D}$ is not prime, $[\,\gamma,\, \gamma\,] = 0$ and since $\mathbf{D}$ is not abelian, but satisfies C1, $\gamma \leq \kappa < 1$ and $[\,\kappa,\, \kappa\,] = 0$.

Let $\Delta = \Delta_{\gamma,\kappa}$ be the congruence on $\mathbf{D}(\gamma)$ generated by

$$\left\{ \left\langle \binom{x}{x}, \binom{y}{y} \right\rangle : x \mathrel{\kappa} y \right\}.$$

Observe that on $\mathbf{D}(\gamma)$, $\kappa_i = (\eta_i : \gamma_i)$, for $i = 0, 1$, since

$$[\,\delta,\, \gamma_i\,] \leq \eta_i \implies [\,\delta \vee \eta_i,\, \gamma_i\,] = [\,\delta,\, \gamma_i\,] \vee [\,\eta_i,\, \gamma_i\,] \leq \eta_i \implies \delta \leq \delta \vee \eta_i \leq \kappa_i$$

by the homomorphism property of the commutator. Furthermore the following relationships hold among the congruences of $\mathbf{D}(\gamma)$:

$$\left.\begin{aligned} \eta_i \prec \gamma_i &\leq \kappa_i \\ \gamma_0 = \gamma_1 &= \eta_0 \vee \eta_1 \\ \Delta \vee \eta_i &= \kappa_0 = \kappa_1 \\ \Delta \wedge \eta_i &= 0 \end{aligned}\right. \qquad \text{for } i = 0, 1.$$

The last of these follows from the fact that $[\gamma, \kappa] = 0$. See [6, theorem 4.9]. It follows from these identities that $\gamma_0/\eta_0 \searrow \eta_1/0 \nearrow \kappa_0/\Delta \searrow \eta_0/0 \nearrow \gamma_1/\eta_1$, and therefore, η_0 and η_1 are atoms of $\operatorname{Con} \mathbf{D}(\gamma)$.

We claim that Δ is a completely meet-irreducible congruence with (unique) cover κ_0. Let $\lambda > \Delta$ and $\lambda \neq \kappa_0$. From the computation above, κ_0 covers Δ, so $\lambda \not\leq \kappa_0$. Since $\kappa_0 = (\eta_0 : \gamma_0)$ we have $[\lambda, \eta_0 \vee \eta_1] = [\lambda, \gamma_0] \not\leq \eta_0$ and therefore $[\lambda, \eta_1] \not\leq \eta_0$. It follows that $\lambda \wedge \eta_1 \neq 0$. But η_1 is an atom, so $\lambda \geq \eta_1$, thus $\lambda \geq \eta_1 \vee \Delta = \kappa_0$ as desired.

Define $\mathbf{A}$ to be $\mathbf{D}(\gamma)/\Delta$. Then $\mathbf{A}$ is subdirectly irreducible with monolith $\beta = \kappa_0/\Delta$. Note that $[\kappa_0, \kappa_0] = [\eta_0 \vee \Delta, \eta_1 \vee \Delta] \leq \Delta$, so $[\beta, \beta] = 0$ on $\mathbf{A}$. Also $\mathbf{A}$ is non-abelian. For if it were abelian, then in $\mathbf{D}(\gamma)$, $[\eta_1, 1] \leq \Delta \wedge \eta_1 = 0$. But then $[\gamma_0, 1] = [\eta_0 \vee \eta_1, 1] \leq \eta_0$ which contradicts the fact that $(\eta_0 : \gamma_0) = \kappa_0 < 1$. Therefore, in $\mathbf{A}$, $[\beta, 1] = \beta$, by C1, verifying (1) of the lemma.

We define $f : \mathbf{A} \to \mathbf{A}$ by $f\big(\binom{x}{y}/\Delta\big) = \binom{x}{x}/\Delta$. As $\Delta \leq \kappa_0$, this is well-defined. That f is an endomorphism satisfying conditions (2) and (3) above is a straightforward verification. □

In order to continue, we need to recall some facts about our modular variety $\mathcal{V}$. There is a ternary term d in the language of $\mathcal{V}$, called the *difference term*, with the following properties:

(1) $\mathcal{V} \models d(x, x, y) = y$
(2) For every $\mathbf{B}$ in $\mathcal{V}$, abelian congruence θ on $\mathbf{B}$, and $b \in B$, $\langle b/\theta, d \rangle$ is a ternary group, denoted $M(\theta, b)$, and for each n-ary term function t, if $t(b_1, b_2, \dots, b_n) = c$ then t is a ternary group homomorphism from $M(\theta, b_1) \times \cdots \times M(\theta, b_n)$ to $M(\theta, c)$.

For the appropriate definitions and proofs see [6, 5.5–5.8].

Lemma 3. *Let $\mathbf{A}$ be an algebra satisfying the conclusions of Lemma 2. There are automorphisms e_0 and e_1 of $\mathbf{A}(\beta)$ given by:*

$$e_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d(x, fx, y) \\ y \end{pmatrix} \qquad \text{and} \qquad e_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ d(x, fy, y) \end{pmatrix}$$

where d is the difference term for $\mathcal{V}$.

Proof. Recall that for any $x, y \in A$, $\binom{x}{y} \in A(\beta) \iff x \;\beta\; y \iff f(x) = f(y) \;\beta\; y$ by (3) of Lemma 2. Suppose $e_0\binom{x}{y} = e_0\binom{u}{v}$. Then $y = v$ and therefore $x \;\beta\; u$, so $f(x) = f(u)$ and all six elements lie in the same ternary group, $M(\beta, x)$. We have $x - f(x) + y = d(x, f(x), y) = d(u, f(u), v) = u - f(x) + y$, so $x = u$. Thus e_0 is injective. Similarly, one can check that

$$e_0^{-1} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} d(x, y, fx) \\ y \end{pmatrix}.$$

That e_0 is a homomorphism follows from the fact that every term, in particular every basic operation, is a ternary group homomorphism on the $M(\beta, x)$–blocks. □

Proof of Theorem 1. Assume that $\mathcal{V}$ has AP and RS, but fails C2. We shall derive a contradiction. Let $\mathbf{A}$, β, e_0 and e_1 be as in Lemmas 2 and 3. Since $\mathcal{V}$ is residually small, there is a maximal essential extension, $\mathbf{E}$, of $\mathbf{A}$ in $\mathcal{V}$ (see [12]). Observe that $\mathbf{E}$ is subdirectly irreducible. Call its monolith μ. Without loss of generality, we may assume that $\mathbf{A} \subseteq \mathbf{E}$ and $\mu \restriction A \supseteq \beta$.

The automorphisms e_0 and e_1 of Lemma 3 are also embeddings of $\mathbf{A}(\beta)$ into $\mathbf{E}^2$. Let us also define $e_2 : \mathbf{A}(\beta) \to \mathbf{E}^2$ to be the identity map. By the amalgamation property (applied twice), there is an algebra $\mathbf{Q}$ in $\mathcal{V}$ and maps $s_j : \mathbf{E}^2 \to \mathbf{Q}$, for $j = 0, 1, 2$, such that $s_0 \circ e_0 = s_1 \circ e_1 = s_2 \circ e_2$. Furthermore, there is a map $r : \mathbf{E} \to \mathbf{E}^2$ given by $r(x) = \binom{x}{x}$. Then $r \restriction A : \mathbf{A} \to \mathbf{A}(\beta)$, and figure 1 commutes.

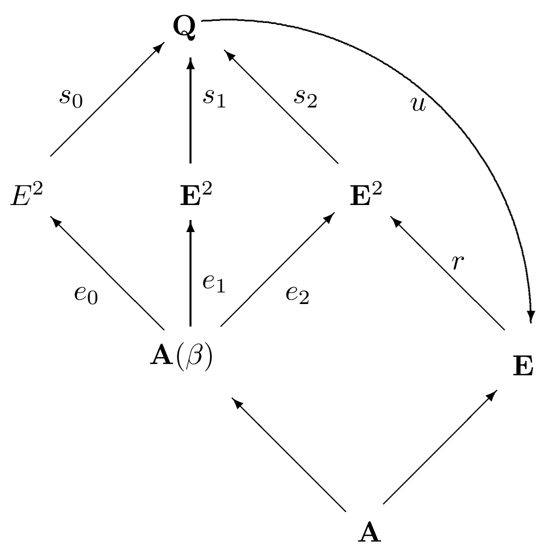

figure 1

Since $\mathbf{E}$ is a maximal essential extension of $\mathbf{A}$, it is an *absolute retract* in $\mathcal{V}$, that is, a retract of each of its $\mathcal{V}$-extensions. Therefore there is a retraction $u : \mathbf{Q} \to \mathbf{E}$ such that $u \circ s_2 \circ r = id_{\mathbf{E}}$. Define

$$\rho_i = \ker(u \circ s_i) \in \operatorname{Con} \mathbf{E}^2 \qquad \text{for } i = 0, 1, 2.$$

Claim. *For each i, if $\rho_i \neq 0$, then either*

$$\rho_i \geq \eta_0 \wedge \mu_1 \text{ and } \rho_i \wedge \eta_1 = 0 \text{ or}$$
$$\rho_i \geq \eta_1 \wedge \mu_0 \text{ and } \rho_i \wedge \eta_0 = 0.$$

Proof of claim. Let us write ρ for ρ_i in the proof of the claim. For any non-zero congruence α on $\mathbf{E}$, $[1_{\mathbf{E}}, \alpha] \supseteq [1_{\mathbf{E}}, \mu] \supseteq [1_{\mathbf{A}}, \beta] = \beta \neq 0_{\mathbf{A}}$; that is, $\mathbf{E}$ is centerless. Therefore, by the homomorphism property, $\mathbf{E}^2$ is centerless. Thus

$$0_{\mathbf{E}^2} \neq [1, \rho] = [\eta_0 \vee \eta_1, \rho] = [\eta_0, \rho] \vee [\eta_1, \rho]$$

so, say $[\eta_0, \rho] \neq 0$. Then $\eta_0 \wedge \rho \neq 0$, implying $\rho \geq \eta_0 \wedge \mu_1$ (since $\eta_0/0 \nearrow 1/\eta_1$.) Furthermore, $\rho \wedge \eta_1 = 0$, for if not, then $\rho \geq \eta_1 \wedge \mu_0$, so $\rho \geq (\eta_0 \wedge \mu_1) \vee (\eta_1 \wedge \mu_0) = \mu_0 \wedge \mu_1$. But choose $(a, b) \in \beta - 0_{\mathbf{A}}$. Recall $\mu \supseteq \beta$ and $f(a) = f(b)$. Since d is a term, it follows that $d(a, fa, a)\ \mu\ d(b, fb, b)$ and therefore $e_i\binom{a}{a} \equiv e_i\binom{b}{b} \pmod{\mu_0 \wedge \mu_1}$, hence modulo ρ as well. But then $u \circ s_i \circ e_i\binom{a}{a} = u \circ s_i \circ e_i\binom{b}{b}$, so by the commutativity of fig. 1, $a = u \circ s_2 \circ r(a) = u \circ s_2 \circ r(b) = b$, which is a contradiction. Therefore we must have $\rho \not\geq \eta_1 \wedge \mu_0$, so $\rho \wedge \eta_1 = 0$. This proves the claim.

We first apply the claim to ρ_2. Certainly, $\rho_2 \neq 0$, in fact, $\binom{x}{y}\ \rho_2\ \binom{z}{z}$, for $z = u \circ s_2\binom{x}{y}$ (since $u \circ s_2\binom{z}{z} = u \circ s_2 \circ r(z) = z$.) Let us assume that $\rho_2 \geq \eta_0 \wedge \mu_1$, and we will derive a contradiction. The case $\rho_2 \geq \eta_1 \wedge \mu_0$ will follow by symmetry. We have $[1, \rho_2] = [\eta_0 \vee \eta_1, \rho_2] \leq \eta_0 \vee (\eta_1 \wedge \rho_2) = \eta_0$, so $[1, \rho_2 \vee \eta_0] \leq \eta_0$ implying $\rho_2 \leq \eta_0$ since $\mathbf{E}$ is centerless. For any $x, y \in E$, $\binom{x}{y} \equiv r \circ u \circ s_2\binom{x}{y} = \binom{z}{z} \pmod{\rho_2}$ for some $z \in E$, whence $x = z$. Thus $\binom{x}{y}\ \rho_2\ \binom{x}{x}\ \rho_2\ \binom{x}{y'}$ and we conclude that $\rho_2 = \eta_0$.

Therefore

$$\eta_0^{\mathbf{A}(\beta)} = e_2^{-1}(\rho_2) = \ker(u \circ s_2 \circ e_2) = \ker(u \circ s_0 \circ e_0) = e_0^{-1}(\rho_0).$$

Observe that for every $a \in A$, $e_0\binom{fa}{a} = \binom{d(fa, fa, a)}{a} = \binom{a}{a}$. Choose $(a, b) \in \beta - 0_{\mathbf{A}}$. Then

$$\begin{pmatrix} fa \\ a \end{pmatrix} \eta_0 \begin{pmatrix} fb \\ b \end{pmatrix} \implies u \circ s_0 \circ e_0 \begin{pmatrix} fa \\ a \end{pmatrix} = u \circ s_0 \circ e_0 \begin{pmatrix} fb \\ b \end{pmatrix} \implies \begin{pmatrix} a \\ a \end{pmatrix} \rho_0 \begin{pmatrix} b \\ b \end{pmatrix}.$$

Now apply the claim to ρ_0. If $\rho_0 \geq (\eta_0 \wedge \mu_1)$ then $\binom{a}{b}\ \rho_0\ \binom{a}{a}\ \rho_0\ \binom{b}{b}$, contradicting $\rho_0 \wedge \eta_1 = 0$. If $\rho_0 \geq (\eta_1 \wedge \mu_0)$ then

$$\eta_0 = e_0^{-1}(\rho_0) \geq e_0^{-1}(\eta_1 \wedge \mu_0) \geq \eta_1 \wedge \beta_0$$

on $\mathbf{A}(\beta)$ which is false, proving the theorem. □

We now turn to our second theorem. In [1, theorem 6] it was proved that if $\mathcal{V}$ is congruence modular, 4-finite and satisfies C2 and R, then $\mathcal{V}$ satisfies AP + RS $\implies$ CEP. Theorem 1 above, eliminates the need to assume C2. The proof below does without the assumption of 4-finiteness. In addition, it provides a considerable simplification of the previous argument. We require one lemma from that paper.

Lemma. *Let $\mathcal{V}$ be a congruence modular variety satisfying AP and RS. Let $\mathbf{A}$ be a subdirectly irreducible member of $\mathcal{V}$, and assume $\mathbf{A}$ is an essential extension of $\mathbf{B}_0 \times \mathbf{B}_1$. Then either $\mathbf{B}_0$ or $\mathbf{B}_1$ is trivial.*

Theorem 4. *Let $\mathcal{V}$ be congruence modular and satisfy R. If $\mathcal{V}$ has AP and RS, then $\mathcal{V}$ has CEP.*

Proof. By Theorem 1, $\mathcal{V}$ has C2. Let $\mathbf{A}, \mathbf{B} \in \mathcal{V}$, with $\mathbf{A} \leq \mathbf{B}$ and θ a completely meet-irreducible congruence on $\mathbf{A}$. It suffices to show that θ extends to $\mathbf{B}$. By R, $[1_{\mathbf{B}}, 1_{\mathbf{B}}] \restriction \mathbf{A} = [1_{\mathbf{A}}, 1_{\mathbf{A}}]$. Thus $\mathbf{A}/[1, 1]$ can be embedded in $\mathbf{B}/[1, 1]$. Suppose first that $\theta \geq [1, 1]$ on $\mathbf{A}$. Since $\mathbf{B}/[1, 1]$ is abelian, it has CEP, so the congruence $\theta/[1, 1]$ (of $\mathbf{A}/[1, 1]$) extends to a congruence $\psi/[1, 1]$ on $\mathbf{B}/[1, 1]$. Then $\psi \in \operatorname{Con} \mathbf{B}$ and $\psi \restriction \mathbf{A} = \theta$.

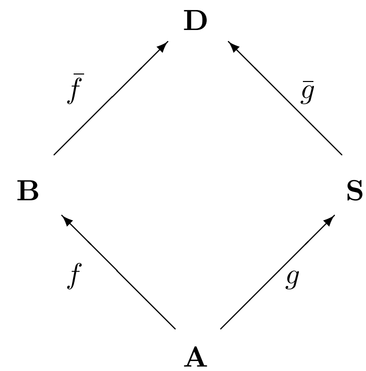

figure 2

So we may assume that $\theta \not\geq [1, 1]$. Let f be the embedding of $\mathbf{A}$ into $\mathbf{B}$. Define $\mathbf{S}$ to be $(\mathbf{A}/\theta) \times \mathbf{A}$ and $g : \mathbf{A} \to \mathbf{S}$ by $g(a) = \binom{a/\theta}{a}$. We apply the amalgamation property to $(\mathbf{A}, \mathbf{B}, \mathbf{S}, f, g)$ yielding $(\mathbf{D}, \bar{f}, \bar{g})$ with $\mathbf{D} \in \mathcal{V}$ (fig. 2.)

Let η_0 and η_1 denote the coordinate projection kernels on $\mathbf{S}$. It suffices to find γ in $\operatorname{Con}\mathbf{D}$ such that $\bar{g}^{-1}(\gamma) = \eta_0$. By the assumptions on θ, $\mathbf{A}/\theta$ is subdirectly irreducible with monolith μ. Thus, there is a congruence μ_0 on $\mathbf{S}$ covering η_0. The commutator respects finite direct products, so $[1_\mathbf{S}, 1_\mathbf{S}] = [1, 1]_0 \wedge [1, 1]_1 \geq \mu_0 \wedge \eta_1$ since $\mathbf{A}/\theta$ is non-abelian. Therefore

$$(*) \qquad \text{for all } \beta \in \operatorname{Con}\mathbf{S} \qquad \beta \not\leq \eta_0 \iff \beta \geq \mu_0 \wedge \eta_1$$

Proof: Certainly, $\eta_0 \geq \beta \geq \mu_0 \wedge \eta_1$ implies $\mu_0 \wedge \eta_1 = \mu_0 \wedge \eta_1 \wedge \eta_0 = 0$, which is false. Conversely, if $\beta \not\leq \eta_0$ then $\beta \vee \eta_0 \geq \mu_0$. Therefore by additivity and C2,

$$\beta \geq [\beta \vee \eta_0, \beta \vee \eta_1] = (\beta \vee \eta_0) \wedge (\beta \vee \eta_1) \wedge [1, 1] \geq \mu_0 \wedge \eta_1.$$

Now let γ be a maximal congruence on $\mathbf{D}$ such that $\bar{g}^{-1}(\gamma) \leq \eta_0^{\mathbf{S}}$. Then γ is completely meet-irreducible. For suppose $\gamma = \bigcap_{j \in J} \gamma_j$ with $\gamma_j > \gamma$, all j. Then $\bar{g}^{-1}(\gamma_j) \not\leq \eta_0$, so by $(*)$, $\bar{g}^{-1}(\gamma_j) \geq \mu_0 \wedge \eta_1$, for all $j \in J$, and therefore

$$\bar{g}^{-1}(\gamma) = \bigcap_{j \in J} \bar{g}^{-1}(\gamma_j) \geq \mu_0 \wedge \eta_1$$

which implies

$$\bar{g}^{-1}(\gamma) \not\leq \eta_0,$$

which is a contradiction.

Therefore $\mathbf{D}/\gamma$ is subdirectly irreducible and $\bar{g}^{-1}(\gamma) = \eta_0 \wedge \delta_1$ for some $\delta \in \operatorname{Con}\mathbf{A}$, by modularity. Then $\mathbf{S}/\bar{g}^{-1}(\gamma) \cong \mathbf{A}/\theta \times \mathbf{A}/\delta$ and we have an induced embedding $\bar{g}/\gamma : \mathbf{A}/\theta \times \mathbf{A}/\delta \to \mathbf{D}/\gamma$. Furthermore, by the maximality of γ, this embedding is essential. Therefore, by the lemma mentioned above, either $\mathbf{A}/\theta$ or $\mathbf{A}/\delta$ is trivial. But $\theta \not\geq [1, 1]$, so we must have $\mathbf{A}/\delta$ trivial, which means $\delta = 1_\mathbf{A}$, and therefore $\bar{g}^{-1}(\gamma) = \eta_0$ as desired. □

Corollary 5. *Let $\mathcal{V}$ be a congruence distributive variety. If $\mathcal{V}$ has AP and RS, then $\mathcal{V}$ has CEP.*

Corollary 6. *Let $\mathcal{V}$ be a congruence modular variety that is 2-finite. If $\mathcal{V}$ has AP and RS, then $\mathcal{V}$ has CEP.*[1]

Proof. If $\mathcal{V}$ is congruence distributive, the commutator reduces to intersection. Thus $\mathcal{V}$ trivially has R and the result follows from Theorem 4. Suppose $\mathcal{V}$ is congruence modular and 2-finite. By theorem 1, $\mathcal{V}$ has C2. By [2, theorem 8] $\mathcal{V}$ has R. Therefore, by Theorem 4, $\mathcal{V}$ has CEP. □

It is well-known that an arbitrary variety has enough injectives if and only if it has AP, RS, and CEP. By the above Corollaries, it follows that any congruence modular, 2-finite variety has enough injectives if and only if it has AP and RS.

As applications we offer the following Corollaries.

Corollary 7. *Let $\mathcal{V}$ be a variety of groups with AP and RS. Then $\mathcal{V}$ is abelian.*

Proof. By Theorem 1, $\mathcal{V}$ satisfies C2. It suffices to show that every subdirectly irreducible group in $\mathcal{V}$ is abelian. Let $\mathbf{G}$ be subdirectly irreducible, and let $\mathbf{M}$ be the normal subgroup corresponding to the monolith. Choose any $a \in M - \{1\}$, and let $\mathbf{A}$ be the subgroup generated by a. Then $\mathbf{A}$ is abelian and $\mathbf{G}$ is an essential extension of $\mathbf{A}$. Therefore, by [2, Theorem 7], $\mathbf{G}$ is abelian. □

Remarks. In [10, page 43] H. Neumann states the following theorem: If $\mathcal{V}$ is a variety of groups with AP, then either every *finite* member of $\mathcal{V}$ is abelian, or else $\mathcal{V}$ is the variety of all groups.

Observe that the argument in Corollary 7 generalizes to any congruence modular variety $\mathcal{V}$ such that

(i) the free algebra on 2 generators is abelian or
(ii) the free algebra on one generator is non-trivial, abelian and has an idempotent element.

Corollary 8. *The variety of all squags is not residually small.*

Proof. A squag is a groupoid $\langle S, \cdot \rangle$ satisfying

$$x \cdot x = x \qquad x \cdot y = y \cdot x \qquad x \cdot (x \cdot y) = y.$$

Every squag is a quasigroup ("squag" is short for "Steiner quasigroup"), so it follows that the variety of squags is congruence modular.

It is implicit in Bruck [3, Theorem 2.1] that the variety of squags has AP. On the other hand, Quackenbush [11, 7.6] showed that this variety does not have CEP. It is easy to see that the variety is 2-finite, in fact the free squag on $\{x, y\}$ has 3 elements: $x, y, x \cdot y$. Therefore, by Corollary 6, the variety of squags is not residually small. □

A variety $\mathcal{V}$ is *directly representable* if it is finitely generated and contains only finitely many finite, directly indecomposable algebras.

[1] Added in Proof: Recently, Keith Kearnes has shown that the assumption in Corollary 6 that $\mathcal{V}$ be 2-finite can be dropped. Thus, any congruence modular variety with AP and RS has CEP.

Corollary 9.

(1) *Let $\mathcal{V}$ be a directly representable variety with AP. Then $\mathcal{V}$ has CEP. In fact, $\mathcal{V}$ is a varietal product $\mathcal{A} \otimes \mathcal{D}$, in which $\mathcal{A}$ is abelian and $\mathcal{D}$ is a discriminator variety.*

(2) *Let $\mathcal{V}$ be a congruence modular, semi-simple, finitely generated variety with AP. Then $\mathcal{V} = \mathcal{A} \otimes \mathcal{D}$ in which $\mathcal{A}$ is abelian and $\mathcal{D}$ is filtral.*

Proof. By [8], every directly representable variety is congruence permutable (hence modular), satisfies C2 and every subdirectly irreducible member is either simple or abelian. In case (2), semi-simplicity implies C2. Now let $\mathcal{V}$ represent the variety in either case. $\mathcal{V}$ is finitely generated, hence 2-finite. $\mathcal{V}$ has C2, hence C1, hence is residually small (Theorem 0). Therefore, by Corollary 6, $\mathcal{V}$ has CEP.

Let $\mathcal{A}$ be the subvariety of $\mathcal{V}$ consisting of all abelian algebras, and let $\mathcal{D}$ be the subvariety generated by all simple, non-abelian algebras. Then by [7, 7.2], $\mathcal{V} = \mathcal{A} \otimes \mathcal{D}$, and $\mathcal{D}$ is a congruence distributive variety, semi-simple and with CEP. Therefore $\mathcal{D}$ is filtral, proving (2). In case (1), $\mathcal{D}$ is congruence permutable as well, so it is a discriminator variety. □

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