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Algorithms for categorical equivalence

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This paper provides an algorithm that, given two finite algebras **A** and **B** each of arbitrary finite similarity type, determines whether or not **A** and **B** are categorically equivalent. Although the algorithm is not practical in general, we consider some conditions on the algebras that improve its performance.

There are several natural equivalence relations that can be imposed upon the space of algebraic structures. At one extreme is isomorphism of algebras. But other relations, such as elementary equivalence and term equivalence, make frequent appearances in the literature.

Given such an equivalence relation, '$\equiv$', it is natural to inquire as to its decidability. That is, does there exist an algorithm that accepts as input two algebras **A** and **B**, (typically with some finiteness conditions imposed) and determines whether or not $\mathbf{A} \equiv \mathbf{B}$ holds. Furthermore, if such an algorithm exists, what kind of bounds can one put on the time needed to perform the computation?

Two algebras **A** and **B** are called *categorically equivalent* if there is an equivalence of the categories $V(\mathbf{A})$ and $V(\mathbf{B})$ that carries **A** to **B**. ($V(\mathbf{A})$ denotes the variety generated by **A**.) We write $\mathbf{A} \equiv_c \mathbf{B}$ to indicate this relationship. Note that there is no expectation that the similarity types of **A** and **B** will be in any way related.

On the face of it, it does not appear likely that the relation '$\equiv_c$' would be decidable, even for finite algebras. However we shall demonstrate in Theorem 3.3 that such an algorithm does exist for finite algebras. The proof is based on a new characterization of categorical equivalence due to R. McKenzie (McKenzie 1996).

In general, our algorithm is not practical, even for small algebras. After presenting the algorithm, we consider conditions that would ensure that a test of $\mathbf{A} \equiv_c \mathbf{B}$ could be

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performed quickly. This is related to the problem of finding, for a finite algebra $\mathbf{A}$, the algebras of smallest cardinality that are categorically equivalent to $\mathbf{A}$.

A knowledge of algebras categorically equivalent to $\mathbf{A}$ can be useful in establishing whether or not $\mathbf{A}$ has a particular algebraic property. For example, suppose $\mathbf{A}$ has k elements and we are given the operation tables of $\mathbf{A}$. We are asked to determine whether $\mathbf{A}$ generates a variety with permuting congruences. A classical theorem states that $\mathbf{A}$ generates such a variety if and only if $\mathbf{A}$ has a Mal'cev term, that is, a ternary term t such that the identities

$$t(x, x, y) = y = t(y, x, x)$$

hold in $\mathbf{A}$. Our search space for such a term can have cardinality as large as k^{k^3}. Thus for even small values of k, the problem is not tractable. On the other hand, the existence of a Mal'cev term is easily seen to be preserved by categorical equivalence. Thus, if we can find an algebra $\mathbf{B}$ such that $\mathbf{A} \equiv_c \mathbf{B}$ and $|B| < k$, we can search in the algebra $\mathbf{B}$ instead and substantially reduce the computational time required. Since many properties of interest are preserved by categorical equivalence, this technique potentially has considerable value.

1. Categorical equivalence

An *algebra* $\mathbf{A} = \langle A, F \rangle$ is a nonvoid set A together with a sequence $F = \langle f_i : i \in I \rangle$ of finitary operations on A. By this we mean that for each $i \in I$, f_i is a function from A^{n_i} to A, for some natural number n_i. The set A is called the *universe* of $\mathbf{A}$ and the components of F are the *fundamental operations* of $\mathbf{A}$. The cardinality of an algebra is the cardinality of its universe. An algebra is *nontrivial* if it has cardinality greater than 1. The *similarity type* of $\mathbf{A}$ is the function $\mathcal{I}$ that assigns to each $i \in I$ the rank n_i of the fundamental operation f_i. An algebra $\mathbf{A}$ is of *finite type* if the index set I is finite.

A *term operation* of $\mathbf{A}$ is any operation on A that can be built by composition from the fundamental operations of $\mathbf{A}$ and the projection operations. A *clone* on a nonvoid set A is any family of operations on A that contains all projection operations and is closed under composition. Thus, the set of term operations of an algebra $\mathbf{A}$ is the smallest clone on A that contains the fundamental operations. $\mathrm{Clo}\,\mathbf{A}$ denotes this clone of term operations and $\mathrm{Clo}_n\,\mathbf{A}$ is the set of n-ary term operations of $\mathbf{A}$.

For an algebra $\mathbf{A}$, the *variety* generated by $\mathbf{A}$, denoted $V(\mathbf{A})$, is the smallest class of algebras that can be obtained from $\mathbf{A}$ by forming homomorphic images, subalgebras, and direct products. By a theorem of Birkhoff, $V(\mathbf{A})$ is the same as the class of all algebras of the same similarity type as $\mathbf{A}$ that satisfy every equation (i.e., identity) that holds in $\mathbf{A}$. The class $V(\mathbf{A})$ can also be considered as a (large) category in a natural way, with the members of $V(\mathbf{A})$ as its objects, and all homomorphisms between algebras as its arrows.

For any variety $V = V(\mathbf{A})$ and positive integer n, the free algebra on n free generators for V always exists. We write $\mathbf{F}_{\mathbf{A}}(n)$ for this algebra. A concrete representation of $\mathbf{F}_{\mathbf{A}}(n)$ is obtained by taking $\mathrm{Clo}_n(\mathbf{A})$ for the universe and letting the fundamental operations be defined in the obvious way. From this we see that if A has k elements, then the cardinality of $\mathbf{F}_{\mathbf{A}}(n)$ is at most k^{k^n}.

In general our notation and terminology follows that in (McKenzie *et. al.* 1987). Thus, for an algebra **A** we write Sub(**A**) for the set of subuniverses of **A**. If $\equiv$ is an equivalence relation on a set S and $x \in S$, then $x/\equiv$ denotes the equivalence class of x with respect to '$\equiv$'. We use ω to represent the set $\{0, 1, 2, 3, \dots\}$.

Two algebras **A** and **B** are called *term-equivalent* if they have the same universe and, for every $n > 0$, $\mathrm{Clo}_n(\mathbf{A}) = \mathrm{Clo}_n(\mathbf{B})$. The algebras **A** and **B** are *weakly isomorphic* if there exists an algebra **C** such that **A** is term-equivalent to **C** and **C** is isomorphic to **B**.

We use the following notation for these relationships.

$\mathbf{A} \simeq \mathbf{B}$	if the algebras are isomorphic;
$\mathbf{A} \equiv_t \mathbf{B}$	if the algebras are term-equivalent;
$\mathbf{A} \equiv_w \mathbf{B}$	if the algebras are weakly isomorphic;
$\mathbf{A} \equiv_v \mathbf{B}$	if the algebras generate the same variety;
$\mathbf{A} \equiv_c \mathbf{B}$	if the algebras are categorically equivalent.

These relations impose quite different restrictions on the specifications of the algebras. Isomorphic algebras must have the same similarity type and have universes of the same cardinality. Term equivalent algebras have the same universes, but the similarity types may differ. Thus, weakly isomorphic algebras require universes of the same size but have no restriction on similarity type. If $\mathbf{A} \equiv_v \mathbf{B}$, then the cardinalities of the universes of the two algebras may be different, but the similarity types must be the same. If $\mathbf{A} \equiv_c \mathbf{B}$, then the algebras may have different similarity types and different sized universes. We note that for an algebra **A** the inclusions $\mathbf{A}/{\equiv_t} \subseteq \mathbf{A}/{\equiv_w} \subseteq \mathbf{A}/{\equiv_c}$ hold.

Let '$\equiv$' denote any of the above equivalence relations. We are interested in the existence of an algorithm that solves the following Basic Problem.

Basic Problem. Given finite algebras **A** and **B** each of finite similarity type, does $\mathbf{A} \equiv \mathbf{B}$ hold?

A precise formulation of this problem must detail the form of the input to the algorithm. We will consider only finite algebras of finite type. Algebras are assumed to be given as a set (the universe) together with the operation tables of the fundamental operations. Thus, the size of the input depends on the size of A and the number and rank of the fundamental operations of **A**.

A problem related to the Basic Problem is the following. Given an algebra **A**, describe the family $\mathbf{A}/{\equiv}$. That is, given an algebra **A**, determine the class of algebras **B** such that $\mathbf{A} \equiv \mathbf{B}$ holds. Certainly, characterizing an object up to isomorphism is a perennial theme in mathematics. A characterization of the class $\mathbf{A}/{\equiv_t}$ is essentially a description of the clone of term operations on **A**.

For categorical equivalence, this sort of characterization seems to be, in general, a difficult task. One might hope to find readily computable properties, which hold for an algebra **A**, that allow for a useful characterization of $\mathbf{A}/{\equiv_c}$. For example, (Bergman and Berman 1996) contains a description of the class $\mathbf{A}/{\equiv_c}$, when **A** is a fixed finite subalgebra-primal, automorphism-primal, or arithmetical and congruence-primal algebra.

Algorithms for deciding the Basic Problem exist for each of the first four equivalences on our list. Although it is hard to find explicit descriptions of these algorithms in the literature, we think it is fair to say that they are familiar to most people working in the field. For completeness, we include a brief discussion of each of them.

To check whether two algebras of cardinality k and finite similarity type $\mathcal{I}$ are isomorphic, one can inspect each bijection to see if it is an isomorphism. For each bijection, the verification time will be of the order of $\sum_{i\in I} k^{n_i}$. The papers (Kučera and Trnková 1984) and (Burris 1995) consider the isomorphism problem for arbitrary algebraic systems.

An algorithm for testing $\mathbf{A} \equiv_v \mathbf{B}$ is presented in (Kalicki 1952). Briefly, the idea is that $\mathbf{B} \in V(\mathbf{A})$ if and only if $\mathbf{B}$ is a homomorphic image of $\mathbf{F_A}(|B|)$. As we have already remarked, this free algebra can be constructed as a subalgebra of $\mathbf{A}^{A^B}$, and whether $\mathbf{B}$ is a homomorphic image can be tested in a manner similar to the procedure outlined in the previous paragraph.

Term-equivalent algebras must have the same universe. If $\mathbf{A} = \langle A, F\rangle$ and $\mathbf{B} = \langle A, G\rangle$, then $\mathbf{A} \equiv_t \mathbf{B}$ is equivalent to

$$F \subseteq \bigcup_{j\le r} \mathrm{Clo}_j(\mathbf{B}) \;\;\&\;\; G \subseteq \bigcup_{j\le r} \mathrm{Clo}_j(\mathbf{A}),$$

where r is the maximum rank of any operation appearing in F or G. That an algorithm for the relation $F \subseteq \mathrm{Clo}_j(\mathbf{B})$ exists follows from the following two assertions.

1 If X is a subset of a finite algebra of finite type, then there is an algorithm that enumerates the members of the subalgebra generated by X, and stops in a finite number of steps.

2 $\mathrm{Clo}_j(\mathbf{B})$ is the subuniverse of $\mathbf{B}^{B^j}$ generated by the j-ary projection functions.

We leave the verification of these two statements to the reader. The second appears as Exercise 1 of Section 4.9 in (McKenzie *et. al.* 1987).

Finally, weak isomorphism composes isomorphism and term-equivalence. Let $\mathbf{A} = \langle A, F\rangle$ and $\mathbf{B} = \langle B, G\rangle$. Then $\mathbf{A} \equiv_w \mathbf{B}$ is equivalent to the existence of a bijection $\varphi\colon A \to B$ such that $F^\varphi \subseteq \mathrm{Clo}(\mathbf{B}) \;\;\&\;\; G^{(\varphi^{-1})} \subseteq \mathrm{Clo}(\mathbf{A})$. Here $F^\varphi = \{\, f^\varphi : f \in F\,\}$, and $f^\varphi(b_1,\dots,b_r) = \varphi f(\varphi^{-1}(b_1),\dots,\varphi^{-1}(b_r))$. Thus one can apply the term-equivalence algorithm to every possible bijection. Since we refer to it later on, we state this last result as a Theorem.

Theorem 1.1. Let $\mathbf{A}$ and $\mathbf{B}$ be finite algebras of finite (but possibly different) similarity type. Then there is an algorithm that determines whether $\mathbf{A} \equiv_w \mathbf{B}$.

Until recently, there has been little attention paid to the relation '$\equiv_c$', or its decidability. It is not obvious when one can extend an assignment $\mathbf{A} \mapsto \mathbf{B}$ to any functor from $V(\mathbf{A})$ to $V(\mathbf{B})$, let alone to an equivalence of categories. In this paper we demonstrate that there is an algorithm for solving the Basic Problem in this case. Our main tool is McKenzie's characterization of varieties that are equivalent as categories. We now outline the basic ideas.

Definition 1.2. Let $\mathbf{A}$ be an algebra of similarity type $\mathcal{I}$, n a positive integer, and σ a unary term operation of $\mathbf{A}$. For any $k \in \omega$, let I_k denote the set of k-ary terms of the similarity type of $\mathcal{I}$.

— For every positive integer p and every sequence $g_1, g_2, \dots, g_n$ of pn-ary operations on A, $(g_1, \dots, g_n)$ denotes the p-ary operation on A^n that maps $(\overline{a}_1, \dots, \overline{a}_p)$ to $(\,g_1(\vec{\mathbf{a}}), g_2(\vec{\mathbf{a}}), \dots, g_n(\vec{\mathbf{a}})\,)$, where $\overline{a}_i = (a_{1i}, \dots, a_{ni}) \in A^n$, and

$$\vec{\mathbf{a}} = (a_{11}, a_{21}, \dots, a_{n1}, a_{12}, \dots, a_{np}) \in A^{pn}.$$

— The *n-th matrix power* of $\mathbf{A}$ is the algebra $\mathbf{A}^{[n]}$ with universe A^n and similarity type indexed by $\bigcup_{p\in\omega}(I_{np})^n$. For each n-tuple of terms $(g_1, \dots, g_n)$, $\mathbf{A}^{[n]}$ has a basic np-ary operation as described above.

— The unary operation σ is *idempotent* if for every $x \in A$, $\sigma\big(\sigma(x)\big) = \sigma(x)$, and σ is *invertible* if for some r there are $t \in \mathrm{Clo}_r(\mathbf{A})$ and $t_1, \dots, t_r \in \mathrm{Clo}_1(\mathbf{A})$ such that, for every $a \in A$, $t\big(\sigma t_1(a), \sigma t_2(a), \dots, \sigma t_r(a)\big) = a$.

— Let σ be an idempotent term operation of $\mathbf{A}$. By $\mathbf{A}(\sigma)$ we denote the algebra with universe $\sigma(A)$ and similarity type indexed by $\bigcup_{p\in\omega} I_p$. For each natural number p and term $g \in I_p$, $\mathbf{A}(\sigma)$ has a fundamental operation $\sigma \circ g\restriction_{\sigma(A)}$.

McKenzie's Theorem. $\mathbf{A} \equiv_c \mathbf{B}$ if and only if there exist a positive integer m and unary term σ such that $\mathbf{B}$ is term-equivalent to an algebra isomorphic to $\mathbf{A}^{[m]}(\sigma)$, with σ invertible and idempotent.

When $\mathbf{A} \equiv_c \mathbf{B}$, the functor $\mathbf{C} \mapsto \mathbf{C}^{[m]}(\sigma)$ realizes (up to weak isomorphism) a categorical equivalence between $V(\mathbf{A})$ and $V(\mathbf{B})$.

Thus we can formulate our Basic Problem for '$\equiv_c$' in the following somewhat stronger form. Given finite algebras $\mathbf{A}$ and $\mathbf{B}$, find (if it exists) a natural number m and term operations σ, t, t_1, … ,t_r with σ idempotent, such that $\mathbf{A}^{[m]} \models x = t(\sigma t_1(x), \dots, \sigma t_r(x))$ and $\mathbf{B} \equiv_w \mathbf{A}^{[m]}(\sigma)$.

There is no à priori upper bound on the size of m in this formulation. Thus, it is not apparent that an algorithm for finding the above terms will exist. Also, McKenzie's Theorem does not provide a bound on the rank r of the term t that serves to invert σ. However, for a finite algebra $\mathbf{A}$ it is obvious that the rank of such a term can be bounded above by the number of distinct unary term operations of $\mathbf{A}^{[m]}$. Thus, $r \le |F_{\mathbf{A}^{[m]}}(1)| = |F_{\mathbf{A}}(m)|^m$. A sharper bound on r is given in Section 3.

We use the following conventions throughout the paper. $\mathbf{A}$ and $\mathbf{B}$ denote finite algebras of size k and ℓ respectively. We reserve σ for an idempotent term and t, $t_1, \dots, t_r$ for terms on $\mathbf{A}^{[m]}$ for which

$$\mathbf{A}^{[m]} \models x = t\big(\sigma t_1(x), \dots, \sigma t_r(x)\big), \tag{1}$$

provided such terms exist. We always let r be the rank of t and we say $t, t_1, \dots, t_r$ are terms that invert σ.

In the next section we consider a special case of the Basic Problem. Given finite algebras $\mathbf{A}$ and $\mathbf{B}$, we provide an algorithm that determines whether or not there is an invertible idempotent term operation σ on $\mathbf{A}$ such that $\mathbf{B} \equiv_w \mathbf{A}(\sigma)$. Then in Section 3 we prove that that if there are m and σ such that $\mathbf{B} \equiv_w \mathbf{A}^{[m]}(\sigma)$, then m can be bounded above by $|B|^{|B|}$. A combination of these two claims provides an algorithm that solves our Basic Problem. Following that, we consider a condition on $\mathbf{B}$ that ensures that we can take $m = 1$. Several applications are presented there.

2. An algorithm for finding σ

Suppose we are given a finite algebra $\mathbf{C}$ of finite similarity type, and a positive integer $s < |C|$. We wish to describe a procedure that will find an invertible idempotent term operation σ on $\mathbf{C}$ with $|\sigma(C)| = s$, if one exists, and halts otherwise.

We assume that the given algebra $\mathbf{C}$ has universe $C = \{0, 1, \ldots, k-1\}$. We view the free algebra $\mathbf{F} = \mathbf{F_C}(1)$ as $\mathrm{Sg}^{\mathbf{C}^k}(x)$, where $x = \langle 0, 1, \ldots, k-1\rangle$. Thus $\mathbf{F}$ has at most k^k distinct members. Each unary term operation g of $\mathbf{C}$ corresponds to a member of C^k and is in $\mathbf{F} = \mathbf{F_C}(1)$ by the correspondence $g \mapsto \langle g(0), \ldots, g(k-1)\rangle$. Consider idempotent terms with range having size s. For each such term, say σ, let $t_1, \ldots, t_r$ denote all members of F that have range contained in $\sigma(C)$. Note that $\sigma t_i = t_i$ for all i, $1 \le i \le r$. Form the subuniverse S of $\mathbf{F}$ generated by all of the t_i. If S contains x, then σ is invertible and we may select an inverting term t for σ that has rank r. Namely, let t be the term used to obtain $x \in S = \mathrm{Sg}^{\mathbf{F}}(\{t_1, \ldots, t_r\})$. If $x \notin S$, then we may conclude that neither σ nor any idempotent unary term operation of $\mathbf{F}$ with range $\sigma(C)$ is invertible.

If we find an invertible idempotent term σ for $\mathbf{C}$, we are still faced with the problem of determining whether or not $\mathbf{C}(\sigma) \equiv_w \mathbf{B}$. From Theorem 6.8 in (McKenzie 1996) we know that if $\mathbf{C}$ has fundamental operations of rank at most n, then $\mathbf{C}(\sigma)$ is term-equivalent to an algebra of finite type having fundamental operations of rank bounded by nr. Therefore, by Theorem 1.1, there is an algorithm to test $\mathbf{B} \equiv_w \mathbf{C}(\sigma)$. We summarize this as a lemma.

Lemma 2.1. Let $\mathbf{C}$ and $\mathbf{B}$ be finite algebras of finite type. There is an algorithm that finds, if it exists, an invertible idempotent term operation σ on $\mathbf{C}$ such that $\mathbf{B} \equiv_w \mathbf{C}(\sigma)$.

If we are given a single algebra $\mathbf{C}$, we can apply the above procedure with s ranging from 2 to $k-1$ to find an invertible idempotent term σ such that $|\sigma(C)|$ is as small as possible. This may provide a useful technique for reducing $\mathbf{C}$ to a categorically equivalent algebra $\mathbf{C}(\sigma)$ of more manageable cardinality.

3. Bounds on m

According to McKenzie's Theorem, we can test $\mathbf{B} \equiv_c \mathbf{A}$ by applying Lemma 2.1 in turn to $\mathbf{A}$, $\mathbf{A}^{[2]}$,$\mathbf{A}^{[3]}$, … . The question is, will this procedure necessarily terminate? In other words, is there an integer M (computable from $\mathbf{A}$ and $\mathbf{B}$) such that $\mathbf{B} \equiv_c \mathbf{A}$, if and only if for some $m \le M$, and some invertible idempotent term σ, $\mathbf{B} \equiv_w \mathbf{A}^{[m]}(\sigma)$. In Corollary 3.2 we show that we can always take $M = |A|^{|B|}$. Later in the section we refine the bound to $|B|^{|B|}$.

Lemma 3.1. Let $\mathbf{A}$ be a finite algebra, n a positive integer, and σ an invertible idempotent term for $\mathbf{A}^{[n]}$. Suppose there exist an integer $m < n$ and m-ary terms $u_{m+1}, \ldots, u_n \in \mathrm{Clo}_m(\mathbf{A})$ such that for each $\overline{b} = (b_1, \ldots, b_n) \in \sigma(A^n)$

$$\overline{b} = (b_1, \ldots, b_m, u_{m+1}(b_1, \ldots, b_m), \ldots, u_n(b_1, \ldots, b_m)).$$

Then there exists an invertible idempotent term σ' for $\mathbf{A}^{[m]}$ such that $\mathbf{A}^{[n]}(\sigma) \equiv_w \mathbf{A}^{[m]}(\sigma') \equiv_c \mathbf{A}$.

Proof. For $\overline{a} = (a_1, \ldots, a_n) \in A^n$ let $\overline{a}^{\downarrow} = (a_1, \ldots, a_m)$ and for $\overline{e} = (e_1, \ldots, e_m) \in A^m$ let $\overline{e}^{\uparrow} = (\overline{e}, u_{m+1}(\overline{e}), \ldots, u_n(\overline{e})) \in A^n$. We have

$$\begin{aligned} &\forall \overline{a} \in A^n \quad (\sigma(\overline{a}))^{\downarrow\uparrow} = \sigma(\overline{a}) \quad \text{ and} \\ &\forall \overline{e} \in A^m \quad \overline{e}^{\uparrow\downarrow} = \overline{e}. \end{aligned} \tag{2}$$

For $f \in \operatorname{Clo}_p \mathbf{A}^{[n]}$ define the p-ary operation f' on A^m by

$$f'(\overline{e}_1, \ldots, \overline{e}_p) = (f(\overline{e}_1^{\uparrow}, \ldots, \overline{e}_p^{\uparrow}))^{\downarrow}$$

for arbitrary $\overline{e}_1, \ldots, \overline{e}_p \in A^m$. We note that if $f = (f_1, \ldots, f_n)$, then $f'(\overline{e}_1, \ldots, \overline{e}_p) = (f_1(\overline{e}_1^{\uparrow}, \ldots, \overline{e}_p^{\uparrow}), \ldots, f_m(\overline{e}_1^{\uparrow}, \ldots, \overline{e}_p^{\uparrow}))$. Thus, $f' \in \operatorname{Clo}_p \mathbf{A}^{[m]}$ since the u_i are in $\operatorname{Clo} \mathbf{A}$.

From this construction we first show for our idempotent term σ, that σ' is also idempotent. For any $\overline{e} \in A^m$ we have $\sigma'\sigma'(\overline{e}) = \sigma'((\sigma(\overline{e}^{\uparrow}))^{\downarrow}) = (\sigma((\sigma(\overline{e}^{\uparrow}))^{\downarrow\uparrow}))^{\downarrow} = (\sigma\sigma(\overline{e}^{\uparrow}))^{\downarrow} = (\sigma(\overline{e}^{\uparrow}))^{\downarrow} = \sigma'(\overline{e})$.

Next, we show that if the terms $t_1, \ldots, t_r$, and t invert σ on $\mathbf{A}^{[n]}$, then $(\sigma t_1)', \ldots, (\sigma t_r)'$ and t' invert σ' on $\mathbf{A}^{[m]}$. Let $\overline{e} \in A^m$ be arbitrary. Then

$$t'\left(\ldots, \sigma'\left((\sigma t_i)'(\overline{e})\right), \ldots\right) = \Big(t\Big(\ldots, \big(\sigma\big((\sigma t_i(\overline{e}^{\uparrow}))^{\downarrow\uparrow}\big)\big)^{\downarrow\uparrow}, \ldots\Big)\Big)^{\downarrow}.$$

This simplifies to

$$\left(t\left(\ldots, \sigma\sigma t_i\left(\overline{e}^{\uparrow}\right), \ldots\right)\right)^{\downarrow} = \left(t\left(\ldots, \sigma t_i\left(\overline{e}^{\uparrow}\right), \ldots\right)\right)^{\downarrow} = (\overline{e}^{\uparrow})^{\downarrow} = \overline{e}$$

as desired.

Finally, let $\mathbf{B} = \mathbf{A}^{[n]}(\sigma)$ and $\mathbf{C} = \mathbf{A}^{[m]}(\sigma')$. We must verify that $\mathbf{B} \equiv_w \mathbf{C}$. We define mappings $\varphi\colon B \to C$ and $\psi\colon C \to B$ by $\varphi(\overline{a}) = \overline{a}^{\downarrow}$, and $\psi(\overline{e}) = \overline{e}^{\uparrow}$. Notice that if $\overline{a} \in B$, then $\overline{a} \in A^n$ and $\sigma(\overline{a}) = \overline{a} = \overline{a}^{\downarrow\uparrow}$. Thus $\varphi(\overline{a}) = \overline{a}^{\downarrow} \in A^m$ and

$$\sigma'(\overline{a}^{\downarrow}) = \sigma(\overline{a}^{\downarrow\uparrow})^{\downarrow} = \sigma(\overline{a})^{\downarrow} = \overline{a}^{\downarrow},$$

so $\varphi(\overline{a})$ is indeed a member of C. On the other hand, for $\overline{e} \in C$, $\overline{e} = \sigma'(\overline{e}) = (\sigma(\overline{e}^{\uparrow}))^{\downarrow}$, and therefore,

$$\psi(\overline{e}) = \overline{e}^{\uparrow} = (\sigma(\overline{e}^{\uparrow}))^{\downarrow\uparrow} = \sigma(\overline{e}^{\uparrow}) \in B.$$

That φ and ψ are inverse to each other follows from equations (2).

Let f be a term operation of $\mathbf{B}$. Then, for some $g \in \operatorname{Clo}(\mathbf{A}^{[n]})$, $f = \sigma g{\restriction_B}$. We claim that $f^{\varphi} = \sigma' \circ (\sigma g)' \in \operatorname{Clo}(\mathbf{C})$ (using the notation preceding Theorem 1.1). To prove this, we assume for clarity that f is unary. Let $\overline{e} \in C$. We have

$$f^{\varphi}(\overline{e}) = \varphi\sigma g(\psi\overline{e}) = (\sigma g(\overline{e}^{\uparrow}))^{\downarrow} = (\sigma(\sigma g(\overline{e}^{\uparrow})))^{\downarrow} = (\sigma((\sigma g)'(\overline{e})^{\uparrow}))^{\downarrow} = \sigma'((\sigma g)'(\overline{e})).$$

Similarly, let h be a unary term operation of $\mathbf{C}$, so that, for some $v \in \operatorname{Clo}_1(\mathbf{A}^{[m]})$, $h = \sigma' v{\restriction_C}$. Define $w(\overline{a}) = (v(\overline{a}^{\downarrow}))^{\uparrow}$. Then w is a term operation of $\mathbf{A}^{[n]}$ and in a manner analogous to the above claim, one can check that $h^{\psi} = \sigma w \in \operatorname{Clo} \mathbf{B}$. Thus $\mathbf{B} \equiv_w \mathbf{C}$. ☐

Corollary 3.2. If $\mathbf{B} = \mathbf{A}^{[n]}(\sigma)$ for an invertible idempotent $\sigma \in \operatorname{Clo}_1(\mathbf{A}^{[n]})$, then there exist $m \leq |A|^{|B|}$ and an invertible idempotent $\sigma' \in \operatorname{Clo}_1(\mathbf{A}^{[m]})$ such that $\mathbf{B} \equiv_w \mathbf{A}^{[m]}(\sigma')$.

Proof. We have $|A| = k$ and we suppose $B = \{\overline{b}_1, \ldots, \overline{b}_\ell\}$. Create an $n \times \ell$ matrix M whose columns are the elements of B. Since A has k elements, M has at most k^{ℓ} distinct rows, say m. Reorder them, putting the distinct rows first. Thus for every $j > m$ there exists an $i \le m$ such that $(\bar{b}_{i1}, \ldots, \bar{b}_{i\ell}) = (\bar{b}_{j1}, \ldots, \bar{b}_{j\ell})$. If we define $u_j(x_1, \ldots, x_m) = x_i$, then the hypotheses of Lemma 3.1 hold. □

With this bound, we have a solution to the Basic Problem.

Theorem 3.3. There is an algorithm that determines given any two finite algebras $\mathbf{A}$ and $\mathbf{B}$ of finite type, whether or not $\mathbf{A} \equiv_c \mathbf{B}$.

Proof. Let $k = |A| \ge |B| = \ell$. We wish to decide if there is an m and an invertible idempotent term σ for $\mathbf{A}^{[m]}$ such that $\mathbf{A}^{[m]}(\sigma) \equiv_w \mathbf{B}$. From the previous Corollary we may restrict to $m \le k^\ell$. For each such m apply Lemma 2.1 to the algebras $\mathbf{C} = \mathbf{A}^{[m]}$ and $\mathbf{B}$. □

Although the Theorem provides an algorithm, an actual computation using this algorithm would not be feasible if the value of m is at all large. Thus, we are interested in conditions on $\mathbf{A}$ and $\mathbf{B}$ that guarantee that a small exponent m for the matrix power can be used if in fact $\mathbf{A} \equiv_c \mathbf{B}$. One approach is to hope that the class $\mathbf{A}/{\equiv_c}$ has a unique smallest member, $\mathbf{M}$ (up to weak isomorphism) and that for every algebra $\mathbf{B} \in \mathbf{A}/{\equiv_c}$ there is an invertible idempotent term σ such that $\mathbf{B}(\sigma) \equiv_w \mathbf{M}$. We could then check $\mathbf{A} \equiv_c \mathbf{B}$ by searching for this σ. We know of several classes of algebras $\mathbf{A}$ with this property. Any subalgebra-primal or automorphism-primal algebra behaves this way, see (Bergman and Berman 1996); also every preprimal algebra, except some of those arising from bounded partial orders, see (Denecke and Lüders 1995; Zádori 1995).

We formalize these considerations with the following definition.

Definition 3.4. An algebra $\mathbf{B}$ is *c-minimal* if for every $\mathbf{A} \equiv_c \mathbf{B}$ there exists an invertible idempotent σ such that $\mathbf{A}(\sigma) \equiv_w \mathbf{B}$.

Clearly, if $\mathbf{B}$ is c-minimal, then $\mathbf{B}$ is minimal-sized in $\mathbf{B}/{\equiv_c}$. Moreover, if $\mathbf{B}$ is also finite and $\mathbf{A} \equiv_c \mathbf{B}$ with $|A| = |B|$, then $\mathbf{A} \equiv_w \mathbf{B}$ (since any idempotent permutation σ is the identity function). Section 4 discusses c-minimality in greater depth. The following example shows that a categorical equivalence class need not possess a c-minimal member.

Example 3.5. Let $A = \{1, 2, 3, 4, 5\}$ and $M = \{1, 2, 3, 4\}$. Let $\mathbf{A}$ be an algebra whose term operations are all operations on A that preserve the two equivalence relations $12|34|5$ and $13|24|5$ and let $\mathbf{M}$ be the algebra whose term operations are all operations on M that preserve the two equivalence relations $12|3|4$ and $1|2|34$. Both $\mathbf{A}$ and $\mathbf{M}$ are congruence-primal and arithmetical. Since the congruence lattices of $\mathbf{A}$ and $\mathbf{M}$ are isomorphic, it follows from (Bergman and Berman 1996, Corollary 4.5) that $\mathbf{M} \equiv_c \mathbf{A}$. Furthermore, it is easy to see that no algebra with fewer than 4 elements can have this congruence lattice. Therefore, $\mathbf{M}$ is of minimal size in $\mathbf{A}/{\equiv_c}$.
Suppose that σ is an invertible idempotent term of $\mathbf{A}$. Then the mapping $\theta \mapsto \theta{\restriction}_{\sigma(A)}$ is an isomorphism of the congruence lattices of $\mathbf{A}$ and $\mathbf{A}(\sigma)$. One easily checks that no proper subset of A can play the role of $\sigma(A)$ here. Therefore σ must be an idempotent permutation of A—which is to say, $\sigma(x) = x$. In particular, $\mathbf{A}(\sigma) \not\equiv_w \mathbf{M}$.

We know of several variations on this example. For example, we can construct algebras $\mathbf{A}$ and $\mathbf{M}$ of cardinalities $2^n + 1$ and $2n$ respectively, and such that $\mathbf{A} \equiv_c \mathbf{M}$ but if $\mathbf{A}^{[m]}(\sigma) \equiv_w \mathbf{M}$ then $m \ge \log_2 n$.

Our approach to a better bound on the parameter m in the relationship $\mathbf{B} \equiv_w \mathbf{A}^{[m]}(\sigma)$ is based on the structure of the subalgebra lattice of $\mathbf{F_B}(1)$. As a bonus, we will also obtain a bound on the rank r of the inverting term.

Let $\mathbf{A}$ be an algebra and suppose $\mathbf{B} = \mathbf{A}^{[n]}(\sigma)$ for a positive integer n and an invertible idempotent $\sigma \in \mathrm{Clo}_1\, \mathbf{A}^{[n]}$. The resulting categorical equivalence $\mathbf{A} \equiv_c \mathbf{B}$ induces a lattice isomorphism between $\mathrm{Sub}\,\mathbf{B}$ and $\mathrm{Sub}\,\mathbf{A}$ given by $S \mapsto \overline{S}$. Thus, $S = \sigma(\overline{S}^{[n]})$. This map extends to direct powers and so for every q we have $\mathrm{Sub}(\mathbf{B}^q)$ and $\mathrm{Sub}(\mathbf{A}^q)$ lattice-isomorphic with $S \mapsto \overline{S}$ for every $S \in \mathrm{Sub}\,\mathbf{B}^q$.

In this case we have

$$S = \big\{ \langle \sigma(\overline{a}^1), \sigma(\overline{a}^2), \ldots, \sigma(\overline{a}^q) \rangle : \langle a_{i1}, \ldots, a_{iq} \rangle \in \overline{S} \text{ for } i = 1, \ldots, n \big\}. \tag{3}$$

Here, we have adopted the following convention. Members of the n-th matrix power of an algebra are written as column vectors while members of the q-th direct power are written as row vectors. Thus, $\overline{a}^j = \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix} \in A^{[n]}$ for $j = 1, \ldots, q$. Also, $\sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_n \end{pmatrix}$ for $\sigma_i \in \mathrm{Clo}_n\, \mathbf{A}$.

Let $\overline{b}^1, \ldots, \overline{b}^q$ be arbitrary members of B and let $S \in \mathrm{Sub}\,\mathbf{B}^q$ be generated by the element $\langle \overline{b}^1, \ldots, \overline{b}^q \rangle$. Form $\overline{T} \in \mathrm{Sub}\,\mathbf{A}^q$ generated by the n elements $\langle b_{i1}, \ldots, b_{iq} \rangle$, for $i = 1, \ldots, n$.

Lemma 3.6. For S and $\overline{T}$ described above, $\overline{S} = \overline{T}$. That is, S and $\overline{T}$ correspond under the lattice isomorphism and thus, $S \equiv_c \overline{T}$.

Proof. First we show $\overline{T} \subseteq \overline{S}$. For this it suffices to show that the generators $\langle b_{i1}, \ldots, b_{iq} \rangle$ of $\overline{T}$ are elements of $\overline{S}$. According to (3), there are elements $\langle a_{i1}, \ldots, a_{iq} \rangle \in \overline{S}$ for $i = 1, \ldots, n$ such that $\overline{b}^j = \sigma \begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$ for $j = 1, \ldots, q$. Working in $\mathbf{A}$ we see that $b_{ij} = \sigma_i(a_{1j}, \ldots a_{nj})$ for all $i \le n$ and $j \le q$. Working in $\mathbf{A}^q$ we get

$$\langle b_{i1}, \ldots, b_{iq} \rangle = \sigma_i\big(\langle a_{11}, \ldots, a_{1q} \rangle, \langle a_{21}, \ldots, a_{2q} \rangle, \ldots, \langle a_{n1}, \ldots, a_{nq} \rangle\big).$$

The q-tuple on the left-hand side is in $\overline{S}$ since the q-tuples on the right are. Hence $\overline{T} \subseteq \overline{S}$.

For the reverse inclusion, we show that $S \subseteq T$, which implies $\overline{S} \subseteq \overline{T}$. Let $T \in \mathrm{Sub}\,\mathbf{B}^q$ be the subuniverse corresponding to $\overline{T}$. Since $\langle b_{i1}, \ldots, b_{iq} \rangle \in \overline{T}$ for $i \le n$, we get

$$\big\langle \sigma(\overline{b}^1), \ldots, \sigma(\overline{b}^q) \big\rangle = \left\langle \sigma \begin{pmatrix} b_{11} \\ \vdots \\ b_{n1} \end{pmatrix}, \ldots, \sigma \begin{pmatrix} b_{1q} \\ \vdots \\ b_{nq} \end{pmatrix} \right\rangle \in T.$$

From the idempotence of σ we have $\sigma(\overline{b}^i) = \overline{b}^i$. Thus, $\langle \overline{b}^1, \ldots, \overline{b}^q \rangle \in T$. □

We introduce one more new concept that may be of interest in its own right.

Definition 3.7. For a finite lattice $\mathbf{L} = \langle L, \wedge, \vee \rangle$ let $h(\mathbf{L})$ denote the least integer with the property that if $1_L = \bigvee S$ for $S \subseteq L$, then there exists a subset Q of S with $|Q| \le h(\mathbf{L})$ such that $1_L = \bigvee Q$.

If $h(\mathbf{L}) = h$, we say that 1_L has *Helly number* h. (See (Garcia and Taylor 1984, page 40) for the use of this term.) We can use this to bound our matrix power.

Theorem 3.8. Suppose $\mathbf{A}$ is a finite algebra, $\mathbf{B} = \mathbf{A}^{[n]}(\sigma)$ for σ an invertible idempotent term, and $\mathbf{F} = \mathbf{F_B}(1)$. Then there exists $m \le h(\operatorname{Sub}\mathbf{F})$ such that $\mathbf{B} \equiv_w \mathbf{A}^{[m]}(\sigma')$ with σ' an invertible idempotent term for $\mathbf{A}^{[m]}$.

Proof. Let $B = \{\overline{b}^1, \dots, \overline{b}^\ell\}$ and $g_i = \langle b_{i1}, \dots, b_{i\ell}\rangle$ for $1 \le i \le n$. The algebra $\mathbf{S} = \operatorname{Sg}^{\mathbf{B}^\ell}\big(\langle\overline{b}^1, \dots, \overline{b}^\ell\rangle\big)$ is isomorphic to $\mathbf{F_B}(1) = \mathbf{F}$. Let $\overline{\mathbf{T}} = \operatorname{Sg}^{\mathbf{A}^\ell}\big\{g_1, \dots, g_n\big\}$. From Lemma 3.6 we have $\mathbf{F} \simeq \mathbf{S} \equiv_c \overline{\mathbf{T}}$. It follows that the subalgebra lattices of $\mathbf{F}$ and $\overline{\mathbf{T}}$ are isomorphic.

In the lattice $\operatorname{Sub}(\overline{\mathbf{T}})$, $\overline{T} = \bigvee_{i=1}^n \operatorname{Sg}^{\overline{\mathbf{T}}}(g_i)$. Therefore, for some $m \le h(\operatorname{Sub}(\overline{\mathbf{T}})) = h(\operatorname{Sub}(\mathbf{F}))$ we have (after reindexing) $\overline{T} = \bigvee_{i=1}^m \operatorname{Sg}^{\overline{\mathbf{T}}}(g_i)$. So for each j with $m+1 \le j \le n$ there exists a term u_j for $\mathbf{A}$ such that $u_j(g_1, \dots, g_m) = g_j$. Thus, the hypotheses of Lemma 3.1 are satisfied. □

This theorem suggests that it would be quite interesting to investigate properties of either $\mathbf{B}$ or $\mathbf{F}$ that are related to the quantity $h(\operatorname{Sub}\mathbf{F})$. In the next section we consider some consequences of the condition $h(\operatorname{Sub}\mathbf{F}) = 1$. We conclude the current section with some lattice-theoretic bounds for $h(\mathbf{L})$ and then some applications of Theorem 3.8.

A crude upper bound for $h(\mathbf{L})$ is $|L|$. The element 1_L in $\mathbf{L}$ is join-irreducible if and only if $h(\mathbf{L}) = 1$. The following Theorem provides several sharper upper bounds on the Helly number. By a *coatom* of $\mathbf{L}$, we mean a maximal member of $\{\, x \in L : x < 1_L \,\}$.

Theorem 3.9. Let $\mathbf{L}$, $\mathbf{L}_1$ and $\mathbf{L}_2$ be finite lattices.

1 $h(\mathbf{L})$ is bounded above by both the length and the number of coatoms of $\mathbf{L}$.
2 If $\mathbf{L}$ is distributive, then $h(\mathbf{L})$ is equal to the number of coatoms of $\mathbf{L}$.
3 If u is the meet of the set of coatoms of $\mathbf{L}$, and if $M = \{\, x \in L : u \le x \,\}$, then $h(\mathbf{M}) = h(\mathbf{L})$.

Proof. For the first claim, let $S = \{s_1, s_2, \dots, s_m\}$ be a minimal subset such that $1 = \bigvee S$. From the minimality, we have a strictly increasing sequence $0 < s_1 < s_1 \vee s_2 < \cdots < s_1 \vee s_2 \vee \cdots \vee s_m$, so $m \le \operatorname{length}(\mathbf{L})$.

Now let the set of coatoms of $\mathbf{L}$ be $\{c_1, \dots, c_k\}$. Since $\bigvee S = 1$, for each $i \le k$ there is $j_i \le m$ such that $s_{j_i} \nleq c_i$. Then $\bigvee\{s_{j_1}, s_{j_2}, \dots, s_{j_k}\} = 1$, so by the minimality of S, $m \le k$.

Suppose that $\mathbf{L}$ is distributive and $\{c_1, \dots, c_k\}$ again denotes the set of coatoms. For $i \le k$ let $c'_i = \bigwedge_{j \ne i} c_j$. Then $c'_i \nleq c_i$ for otherwise

$$c_i = c_i \vee c'_i = c_i \vee \bigwedge_{j\ne i} c_j = \bigwedge_{j \ne i} (c_i \vee c_j) = 1$$

which is false. Then the set $\{c'_1, c'_2, \dots, c'_k\}$ is a minimal set with join 1. The elements of this set are pairwise distinct. For if $i \ne j$ then $c'_i \nleq c_i$, while $c'_j \le c_i$. Therefore, $h(\mathbf{L}) \ge k$. Combining this with part 1 of the Theorem yields part 2.

Now let $\mathbf{L}$ be an arbitrary finite lattice and let u and M be as in part 3. For $x \in L$ let $\gamma(x)$ denote the set of coatoms above x. It is easily checked that $x \le \bigwedge \gamma(x)$. Moreover, for $a_1, \dots, a_n \in L$ we have $1_L = a_1 \vee \cdots \vee a_n$ if and only if $\varnothing = \gamma(a_1) \cap \cdots \cap \gamma(a_n)$. Thus,

$1_L = a_1 \vee \cdots \vee a_n$ if and only if $1_L = \big(\bigwedge \gamma(a_1)\big) \vee \cdots \vee \big(\bigwedge \gamma(a_n)\big)$. This latter join is also in M, so $h(\mathbf{L}) = h(\mathbf{M})$, proving 3. □

Note that if $\mathbf{G}$ is any algebra and we take $L = \operatorname{Sub}\mathbf{G}$ in Theorem 3.9(3), then the point u represents the Frattini subalgebra of $\mathbf{G}$. Also, in the case that $\operatorname{Sub}\mathbf{G}$ is distributive, we see that $h(\operatorname{Sub}\mathbf{G})$ is equal to the number of maximal subalgebras of $\mathbf{G}$.

In Corollary 3.2 we obtained the bound $m \leq |A|^{|B|}$. We can eliminate the dependence on $\mathbf{A}$ here. Let $\mathbf{F} = \mathbf{F_B}(1)$. Then by Theorems 3.8 and 3.9

$$m \leq h(\operatorname{Sub}\mathbf{F}) \leq \operatorname{length}(\operatorname{Sub}\mathbf{F}) \leq |F| \leq |B|^{|B|}.$$

The Helly number of 1_L can be used to bound another quantity that appears in our investigation of categorical equivalence for algebras, namely the rank r of the inverting term t in equations (1).

Proposition 3.10. Let σ be an invertible idempotent term on $\mathbf{A}^{[m]}$. Then there is an r-ary term t with $r \leq h\big(\operatorname{Sub}(\mathbf{F_A}(m))\big)$ and unary terms $t_1, \ldots, t_r$ such that $\mathbf{A}^{[m]}$ satisfies the identity $x = t(\sigma t_1(x), \ldots, \sigma t_r(x))$.

Proof. Take $\mathbf{C} = \mathbf{A}^{[m]}$ and $\mathbf{F} = \mathbf{F_C}(1)$. Since σ is assumed to be invertible on $\mathbf{C}$, there is a q-ary term t' and unary terms $t_1, \ldots, t_q$ so that $\mathbf{F} \vDash x = t'(\sigma t_1(x), \ldots, \sigma t_q(x))$. This is equivalent to the assertion $F = \operatorname{Sg}^{\mathbf{F}}(\sigma t_1) \vee \cdots \vee \operatorname{Sg}^{\mathbf{F}}(\sigma t_q)$ in the lattice $\operatorname{Sub}\mathbf{F}$. Since the algebras $\mathbf{F}$ and $\mathbf{F_A}(m)^{[m]}$ are weakly isomorphic and $\mathbf{F_A}(m)^{[m]} \equiv_c \mathbf{F_A}(m)$, we obtain $\operatorname{Sub}(\mathbf{F}) \simeq \operatorname{Sub}(\mathbf{F_A}(m))$. It follows that for some $r \leq h\big(\operatorname{Sub}(\mathbf{F_A}(m))\big)$ we have (after reindexing) that $F = \operatorname{Sg}^{\mathbf{F}}(\sigma t_1) \vee \cdots \vee \operatorname{Sg}^{\mathbf{F}}(\sigma t_r)$. Therefore, there is an r-ary term t such that the equation $x = t(\sigma t_1(x), \ldots, \sigma t_r(x))$ holds in $\mathbf{F}$, hence in $\mathbf{C}$. □

This Proposition has at least one noteworthy consequence. Suppose that $\mathbf{G} = \mathbf{F_A}(m)$ has the property that the complement of each free generator is a subuniverse. For example, in varieties of lattices, the generators are always doubly irreducible, so this property holds. Then the coatoms of $\operatorname{Sub}\mathbf{G}$ would be exactly those subuniverses. Thus $r \leq h(\operatorname{Sub}\mathbf{G}) \leq m$.

If σ is an invertible idempotent term for $\mathbf{C}$ having r for the rank of an inverting term, then r and $|\sigma(C)|$ are related by the inequality $|\sigma(C)|^r \geq |C|$. Thus, $|\sigma(C)| \geq |C|^{1/h(\operatorname{Sub}\mathbf{F})} = |A|^{m/h(\operatorname{Sub}\mathbf{F})}$. This bound may be used to limit the range of the parameter s in the algorithm for finding σ as discussed at the end of Section 2.

4. On c-minimal algebras

For any finite lattice $\mathbf{L}$, the condition $h(\mathbf{L}) = 1$ is equivalent to the assertion that the element 1_L is join-irreducible. When applied to the subalgebra lattice of a free algebra, we obtain a sufficient condition for c-minimality.

Theorem 4.1. Let $\mathbf{B}$ be a finite algebra, $\mathbf{F} = \mathbf{F_B}(1)$, and suppose that F is join-irreducible in $\operatorname{Sub}\mathbf{F}$. Then $\mathbf{B}$ is c-minimal.

Proof. Let $\mathbf{A} \equiv_c \mathbf{B}$. As we remarked above, $h(\operatorname{Sub}\mathbf{F}) = 1$, so by McKenzie's Theorem and Theorem 3.8 there exists an invertible idempotent σ on $\mathbf{A}$ such that $\mathbf{A}(\sigma) \equiv_w \mathbf{B}$. □

In this section we give an algorithm which decides, for a given finite algebra $\mathbf{B}$, if F is join-irreducible in $\operatorname{Sub}(\mathbf{F})$. We also present some conditions on $\mathbf{B}$ that guarantee that $\mathbf{F}$

has this property. In Examples 4.8 and 4.9 we show that while the condition $h(\operatorname{Sub}\mathbf{F}) = 1$ is sufficient for c-minimality, it is not necessary.

We view each element $g \in F$ as a unary operation on B. Let

$$N = \{g \in F \mid g \text{ is not a permutation of } B\}.$$

Lemma 4.2. *F is join-irreducible in $\operatorname{Sub}\mathbf{F}$ if and only if N is a subuniverse of $\mathbf{F}$.*

Proof. If $g \in F$ is a permutation of B, then since $|B| = \ell$ we have that $g^{\ell!}(x) = x$ for all $x \in B$. Thus, $\operatorname{Sg}^{\mathbf{F}}(g) = F$. On the other hand, if g is not a permutation of B, then $\operatorname{Sg}^{\mathbf{F}}(g)$ is a proper subuniverse of $\mathbf{F}$ since if $i, j \in B$ are such that $g(i) = g(j)$, then $f(i) = f(j)$ for all $f \in \operatorname{Sg}^{\mathbf{F}}(g)$.

Suppose F is join-irreducible with F_0 the unique subuniverse covered by F in $\operatorname{Sub}\mathbf{F}$. From the previous paragraph we conclude that $F_0 = N$, so N is a subuniverse of $\mathbf{F}$. Conversely, since every proper subuniverse of $\mathbf{F}$ is contained in N, if N is itself a subuniverse, it must be the largest one. □

As an application of Lemma 4.2, we discover that every finite multi-unary algebra is c-minimal. In this case, it is quite easy to check that N is always a subuniverse.

To decide for a given algebra $\mathbf{B}$ if the universe F is join-irreducible in $\operatorname{Sub}\mathbf{F}$, we can do the following. First generate $\mathbf{F}$ as a subuniverse of $\mathbf{B}^{|B|}$. Note that for $B = \{0, 1, \dots, \ell-1\}$ the algebra $\mathbf{F}$ is generated by the single ℓ-tuple $\langle 0, 1, \dots, \ell-1\rangle$. This gives F as a collection of ℓ-tuples. Let N be the subset of F consisting of those ℓ-tuples that are not permutations of B. Form the subuniverse of $\mathbf{B}^{\ell}$ generated by N. This subuniverse is N itself if and only if F is join-irreducible.

A simple condition on $\mathbf{B}$ that forces F to be join-irreducible in $\operatorname{Sub}\mathbf{F}$ is $f(x, \dots, x) = x$ for every fundamental operation f of $\mathbf{B}$. For in this case $F = \{x\}$ and $N = \varnothing$. Such $\mathbf{B}$ are often called *idempotent* in the literature. The c-minimality of idempotent algebras is demonstrated in (Ježek 1982). See also (McKenzie 1996, Theorem 6.3).

More generally, if the free algebra $\mathbf{F}$ is such that $\operatorname{Sub}(\mathbf{F})$ forms a chain, then $\mathbf{B}$ is c-minimal. There are several well-known examples of this phenomenon, such as (bounded) lattices, (bounded) semilattices, Boolean algebras, and groups of prime exponent. An easily checked sufficient condition for $\operatorname{Sub}(\mathbf{F})$ to be a chain is that $|F| - |C| \leq 2$, where C is the subuniverse consisting of all constant term operations of $\mathbf{F}$.

Definition 4.3. An algebra $\mathbf{A}$ is *term-minimal* if $\mathbf{A}$ is finite, nontrivial, and every idempotent unary term operation of $\mathbf{A}$ is either a constant or the identity operation.

Term-minimal algebras play an important role in the study of minimal varieties, i.e., equationally complete equational classes. See (Szendrei 1994; Kearnes and Szendrei 1996)

If in the definition of term-minimal algebra we replace "term" by "polynomial" we obtain the definition of an *E-minimal algebra* and if we then delete "idempotent" we have the definition of a *minimal* algebra. Thus, every E-minimal or minimal algebra is term-minimal. E-minimal algebras and especially minimal algebras are central to the development of tame congruence theory (Hobby and McKenzie 1988). It is known that the type set (in the sense of tame congruence theory) of a finite algebra is preserved under categorical equivalence. See (Denecke 1990) for minimal algebras and (McKenzie 1996, Theorem 6.8) for the general case. The property of being (term) minimal is not preserved by categorical equivalence since the matrix square of any nontrivial algebra is not term-minimal. We now show that every term-minimal algebra is c-minimal.

We will need the following easily proved result in our investigations of categorical equivalence for term-minimal algebras.

Lemma 4.4. Let f be a unary operation on a finite set A and suppose there exist $a, b \in A$ such that $\{f(a), f(b)\} = \{a, b\}$. Then there is an integer r such that f^r is idempotent, $f^r(a) = a$, and $f^r(b) = b$.

Theorem 4.5. Let $\mathbf{A}$ be a term-minimal algebra and let $\mathbf{F}$ denote $\mathbf{F}_{\mathbf{A}}(1)$. Then F is join-irreducible in $\operatorname{Sub}\mathbf{F}$.

Proof. By Lemma 4.2, it suffices to show that the set N consisting of members of F that are not permutations of A is a subuniverse of F. We assume the contrary. Thus, the subuniverse generated by N contains permutations. This in turn, implies

$$\forall i \neq j \in A \;\; \exists g \in N \;\text{ such that } g(i) \neq g(j). \tag{4}$$

Let $G = \langle V, E \rangle$ be the directed graph whose vertices are all unordered pairs of distinct elements of A and whose edges are all $(\{i, j\}, \{i', j'\})$ for which there is a $g \in N$ such that $\{g(i), g(j)\} = \{i', j'\}$.

From (4) it follows that the directed graph G has the property that each vertex is the initial vertex of an edge. Since V is finite, G has a cycle $C = \langle v_0, \ldots, v_{s-1} \rangle$, with the edge from v_k to v_{k+1} labeled by $g_k \in N$. (The case $s = 1$ would correspond to a loop at v_0.) Let $g = g_{s-1} \circ \cdots \circ g_0$. Note that g is not one-to-one. Suppose $v_0 = \{i, j\}$. Then $\{g(i), g(j)\} = \{i, j\}$. But then from Lemma 4.4 we see that there exists an integer r for which the operation g^r is idempotent. Since $\mathbf{A}$ is term-minimal this is impossible. So the universe generated by N contains no permutations. □

Corollary 4.6. Every term-minimal algebra is c-minimal.

Since every 2-element algebra is term-minimal we obtain the following.

Corollary 4.7. Let $\mathbf{A}$ and $\mathbf{B}$ each be 2-element algebras. Then $\mathbf{A} \equiv_c \mathbf{B}$ if and only if $\mathbf{A} \equiv_w \mathbf{B}$.

The manuscript (Lüders 1996) contains a description of the equivalence classes of $\equiv_c$ for every 2-element algebra.

We close with two examples to show that the converse of Theorem 4.1 does not hold, in other words, there are c-minimal algebras $\mathbf{B}$ such that $h(\operatorname{Sub}\mathbf{F}_{\mathbf{B}}) > 1$.

Example 4.8. Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be arbitrary algebras. We write $\mathbf{B}_1 \otimes \mathbf{B}_2$ to denote the algebra with universe $B_1 \times B_2$ and with n-ary term operations $f_1 \times f_2$ as f_i ranges through $\operatorname{Clo}_n(\mathbf{B}_i)$, for $i = 1, 2$. Here $f_1 \times f_2$ operates componentwise on $B_1 \times B_2$. The algebra $\mathbf{B}_1 \otimes \mathbf{B}_2$ is unique up to term-equivalence. See (Taylor 1973) for a complete development of the properties of this construction, which corresponds to the product in the category of clones. Note that despite the similarity in notation, $\mathbf{B} \otimes \mathbf{B}$ is unrelated to $\mathbf{B}^{[2]}$.

Claim. If $\mathbf{A} \equiv_c \mathbf{B}_1 \otimes \mathbf{B}_2$ then there are algebras $\mathbf{A}_i \equiv_c \mathbf{B}_i$, for $i = 1, 2$, such that $\mathbf{A} \simeq \mathbf{A}_1 \otimes \mathbf{A}_2$.

Proof. It is not hard to check that $(\mathbf{B}_1 \otimes \mathbf{B}_2)^{[n]} \equiv_w \mathbf{B}_1^{[n]} \otimes \mathbf{B}_2^{[n]}$ and that if $\sigma_1 \times \sigma_2$ is an invertible idempotent term of $(\mathbf{B}_1 \otimes \mathbf{B}_2)$, then $(\mathbf{B}_1 \otimes \mathbf{B}_2)(\sigma_1 \times \sigma_2) \equiv_w \mathbf{B}_1(\sigma_1) \otimes \mathbf{B}_2(\sigma_2)$.

Now suppose that $\mathbf{A} \equiv_c \mathbf{B}_1 \otimes \mathbf{B}_2$. Then for some $m > 0$ and invertible idempotent term $\bar{\sigma}$,

$$\mathbf{A} \equiv_w (\mathbf{B}_1 \otimes \mathbf{B}_2)^{[m]}(\bar{\sigma}) \equiv_w \big(\mathbf{B}_1^{[m]} \otimes \mathbf{B}_2^{[m]}\big)(\bar{\sigma}) \equiv_w \mathbf{B}_1^{[m]}(\bar{\sigma}_1) \otimes \mathbf{B}_2^{[m]}(\bar{\sigma}_2).$$

The last equivalence in the sequence comes from the fact that every term, such as $\bar{\sigma}$, on a varietial product is a product of terms on the factors. We now obtain $\mathbf{A} \simeq \mathbf{A}_1 \otimes \mathbf{A}_2$ where $\mathbf{A}_i \equiv_t \mathbf{B}_i^{[m]}(\bar{\sigma}_i)$, for $i = 1, 2$.

Now suppose that $\mathbf{B}_1$ and $\mathbf{B}_2$ are c-minimal, (for example, they may satisfy the hypothesis of Theorem 4.1), and suppose $\mathbf{B} = \mathbf{B}_1 \otimes \mathbf{B}_2$. Then the Claim implies that $\mathbf{B}$ is c-minimal as well. However, if $\mathbf{F} = \mathbf{F_B}(1)$, then we will not have F join-irreducible in $\operatorname{Sub}\mathbf{F}$. In fact, $\operatorname{Sub}\mathbf{F} \simeq \operatorname{Sub}(\mathbf{F}_{\mathbf{B}_1}) \times \operatorname{Sub}(\mathbf{F}_{\mathbf{B}_2})$. For a concrete example, consider the Abelian groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$, both of which satisfy Theorem 4.1. We have $\mathbb{Z}_6 \equiv_w \mathbb{Z}_2 \otimes \mathbb{Z}_3$, but of course the Frattini subgroup of $\mathbb{Z}_6$ is trivial.

Example 4.9. As another example, let $B = \{1,2,3,4\}$, $B_1 = \{1,2\}$ and $B_2 = \{3,4\}$. Let S be the set of all operations on B that preserve the subsets B_1 and B_2. Then $\mathbf{B} = \langle B, S\rangle$ is a subalgebra-primal algebra, with exactly two nonempty proper subuniverses, B_1 and B_2. It follows from (Bergman and Berman 1996, Theorem 3.1) that $\mathbf{B}$ is c-minimal.
On the other hand, $h(\operatorname{Sub}\mathbf{F}) \neq 1$. To see this, consider the two unary operations f, g and any binary operation b given by the following tables, where each '$*$' denotes an arbitrary element of B.

	f	g
1	1	1
2	1	2
3	3	3
4	4	3

b	1	2	3	4
1	1	2	$*$	$*$
2	2	1	$*$	$*$
3	$*$	$*$	3	4
4	$*$	$*$	4	3

We see that $b(f(x), g(x)) = x$. Since neither f nor g are permutations, it follows from Lemma 4.2 that F is not join-irreducible in $\operatorname{Sub}\mathbf{F}$. Note that $\mathbf{B}$ is simple, so it certainly can not be a product as in the previous example.

Problem. Characterize all (finite) c-minimal algebras.

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