Fibonacci meets Erdős-Ko-Rado

Steve Butler

Drake Mathematics Club
7 October 2011
Leonardo Pisano Bigollo
aka Leonardo of Pisa
aka Leonardo Bonacci
aka Leonardo Fibonacci
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c. 1170 AD – c. 1250 AD
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Introduced decimal notation for working with numbers and wrote the first algebra textbook *Liber Abaci* (1202 AD).
Domino tilings of $2 \times n$

<table>
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<th>$n$</th>
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Let $Q_n$ be the number of tilings of $2 \times n$ with dominoes.
Recurrence relation

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$$Q_n = Q_{n-1} + Q_{n-2}$$
The $Q_n$ are (almost) the Fibonacci numbers. We can now compute the first few terms.

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Fun fact: $Q_n \approx \frac{\varphi^{n+1}}{\sqrt{5}}$ where $\varphi = \frac{1 + \sqrt{5}}{2}$
Erdős-Ko-Rado

Paul Erdős, Chao Ko and Richard Rado were three twentieth century mathematicians.

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Another roof, another proof. — Paul Erdős
Paul Erdős was in *N is a number* with Alec Guinness

Alec Guinness was in *Lovesick* with David Strathairn

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Erdős Number

A mathematician’s Erdős number measures the distance in collaborations to Erdős.

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- Erdős has an Erdős number of 0.
- Someone who wrote a paper with Erdős has an Erdős number of 1.
- Someone who wrote a paper with Erdős number of 1 has an Erdős number of 2.
- … and so on …
Paul Erdős wrote a paper with Mark Kac
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Mark Kac wrote a paper with Wayne Barrett
Paul Erdős wrote a paper with Mark Kac

Mark Kac wrote a paper with Wayne Barrett

Wayne Barrett wrote a paper with Jason Grout
Given a collection of objects a subset is a collection of some of these objects. For example, we can take students to be our objects and group them by what clubs they belong to.
Two subsets *intersect* if they have at least one object in common. Two subsets *do not intersect* if they have no object in common.
**Goal:** Given $n$ objects what is the maximum number of subsets we can make so that each subset has $k$ objects and every pair of subsets intersects?
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Obvious thing: Pick one object and take all subsets with \( k \) objects that contain the fixed object. Total number of such subsets is the number of ways to pick \( k - 1 \) objects out of the \( n - 1 \) objects that remain, this number is denoted \( \binom{n-1}{k-1} \).
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Erdős-Ko-Rado Theorem
If \( n \geq 2k \), you cannot do better than doing the obvious thing.
An Erdős-Ko-Rado problem is one where we have a collection of objects with a notion of intersection and try to find the maximal family.
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We will say that two domino tilings intersect if they have at least one tile in the same location in both tilings (i.e., if they were laid on top of one another).
Question: What is the size of the largest maximal collection of intersecting $2 \times n$ domino tilings? (Recall that there are a total of $Q_n$ tilings of $2 \times n$.)
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**Obvious thing:** Take all the $2 \times n$ tilings which start with a vertical tile. There are $Q_{n-1}$ such tilings.
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Theorem (Butler-Horn-Tressler)
You cannot do better than doing the obvious thing.
Sketch of proof I

Start by taking all of the tilings and group them into those that start with a vertical tile (these will form $S$) and those that start with horizontal tiles (these will form $T$).
Sketch of proof II

Join two tilings with an edge whenever they do not intersect.
Sketch of proof III

For every tiling in $\mathcal{T}$ we can pair it with a tiling in $S$. This is done by breaking up tilings and using the rule as indicated below.

\[
\begin{align*}
\text{2k vertical} & \rightarrow \text{k horizontal} \\
\text{2k - 1 vertical} & \rightarrow \text{k horizontal}
\end{align*}
\]
Given an intersecting family let it consist of $S \cup T$ with $S \subseteq S$ and $T \subseteq T$. 
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**Erdős-Ko-Rado Theorem**

For subsets, if $n$ is sufficiently large then we cannot do better than doing the obvious thing (i.e., fix $t$ elements and take all subsets of size $k$ that contain those $t$ elements).
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What about for domino tilings?

*Obvious*  

*Better*
Fault free tilings

Challenge: Find a fault free tiling of $6 \times 6$.

Tiling with faults

Fault free tiling
Fault free tilings

Tiling with faults

Fault free tiling

**Challenge:** Find a fault free tiling of $6 \times 6$. 