Finding patterns avoiding many monochromatic constellations

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Can we color the integers \([n] = \{1, 2, \ldots, n\}\) with colors \text{RED} and \text{BLUE} so that there is no three term arithmetic progression all red or all blue?

**Theorem (van der Waerden)**

For fixed \(k\) and \(r\), any coloring of \([n]\) with \(n \geq W(r, k)\) using \(r\) colors contains an arithmetic progression of length \(k\).

**Corollary**

For fixed \(k\) and \(r\), there is a \(\gamma > 0\) so that any coloring of \([n]\) using \(r\) colors contains at least \((\gamma + o(1)) n^2\) different monochromatic \(k\)-term arithmetic progressions.
How sparsely can we pack three term arithmetic progressions into a two coloring of $[n]$?

What is the minimal $\gamma$ so that a coloring of $[n]$ with two colors contains

$$(\gamma + o(1)) n^2$$

three term arithmetic progressions?

First guess: Color randomly. $\gamma = \frac{1}{16}$

(There are $\approx n^2/4$ three term arithmetic progressions, in a random coloring a fixed progression has probability of 1/4 of being monochromatic so we expect $\approx n^2/16$.)
A pattern which beats random!

This consists of 12 blocks with relative sizes:

For this coloring we have

$$\gamma = \frac{117}{2192} \approx 0.05337591 \ldots < \frac{1}{16} = 0.0625.$$  

(Found independently by B-C-G and Parrilo-Robertson-Saracino.)

- How did we find this pattern? How did we calculate $\gamma$?
- Why do we think this might be the best pattern?
Constellations

A constellation pattern is a collection of rationals \( q_i \in [0, 1] \) which includes 0 and 1. A constellation is a scaled translated realization in \([n]\).

\[
\begin{align*}
\{0, \frac{1}{2}, 1\} & \leftrightarrow \text{three term AP} \\
\{0, \frac{1}{3}, \frac{2}{3}, 1\} & \leftrightarrow \text{four term AP} \\
\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\} & \leftrightarrow \text{five term AP} \\
\{0, \frac{1}{3}, 1\} & \leftrightarrow \text{solutions to } x + 2y = 3z \\
\{0, \frac{2}{5}, 1\} & \leftrightarrow \text{solutions to } 2x + 3y = 5z \\
\{0, \frac{a}{a+b}, 1\} & \leftrightarrow \text{solutions to } ax + by = (a + b)z
\end{align*}
\]
Problem

Given a constellation pattern, find a small (ideally, smallest) $\gamma$ so that there is a two coloring of $[n]$ with

$$(\gamma + o(1)) n^2$$

monochromatic constellations, and also give the coloring.

As an example we will use the constellation $\{0, \frac{1}{3}, 1\}$. 
Observation
If we are in a coloring of \([n]\) which minimizes the number of monochromatic constellations, then switching the color of any single entry will not cause the number of monochromatic constellations to decrease.

Perturbate to find (local) minimal colorings
Given a two coloring of \([n]\).

- Test to see if changing the color of a given number decreases the number of monochromatic constellations. If it does, change it.
- Go to the next number.
- Stop when changing the coloring of any given number does not decrease the number of monochromatic solutions.
There are two major decisions in applying this procedure:

- What coloring do we begin with?
- How do we go to the next element (i.e., sweep back and forth, random, etc.)?

When we stop we will be at a local minimal, but this might be far from the global minimal. There are several ways to deal with this.

- Run the experiment many times with varying starting patterns and rules for moving to the next element and take the best one that is produced.
- Run the experiment, when it stops hit it with a small perturbation and then keep running it. Repeat several times.
We can test this procedure on Schur triples \((x + y = z)\) for which it is known that there are at least \(\frac{1}{22} n^2 + O(n)\) monochromatic solutions and that the best coloring that achieves this is a block coloring with three blocks with relative sizes 4-6-1. (Robertson-Zeilberger and Schoen)
Now we look at runs for the constellation \( \{0, \frac{1}{3}, 1\} \).
Some more runs for the constellation \( \{0, \frac{1}{3}, 1\} \).
Looking at these runs, we see that they all seem to go to a pattern with 18 blocks.

Doing a run on [25000] we get block sizes


These block sizes are approximations for the relative block sizes in the locally optimal coloring. The next step is to find the correct relative block sizes, and the corresponding $\gamma$. 
Calculating $\gamma$ given a block pattern

Given a constellation pattern, $q_i$, let $D$ be the smallest common denominator. Let $f : [0, 1] \to \{\pm 1\}$ by scaling the block pattern to the interval $[0, 1]$ and sending blue to 1 and red to $-1$. Then

$$\gamma = \begin{cases} \frac{\alpha}{2D} n^2 + O(n) & \text{if constellation is symmetric,} \\ \frac{\alpha}{D} n^2 + O(n) & \text{if constellation is not symmetric.} \end{cases}$$

where

$$\alpha = \int_0^1 \int_0^1 \left( \prod_{i} \frac{1 + f(q_i x + (1 - q_i)y)}{2} \right. $$

$$\left. + \prod_{i} \frac{1 - f(q_i x + (1 - q_i)y)}{2} \right) dy \, dx.$$
Looking more closely at $\alpha$

The function that we are integrating over to calculate $\alpha$,

$$g(x, y) = \prod_i \frac{1 + f(q_i x + (1 - q_i)y)}{2} + \prod_i \frac{1 - f(q_i x + (1 - q_i)y)}{2}$$

is a 0-1 indicator function which indicates monochromatic constellations inside of $[0, 1]$.

The function $g(x, y)$ can only change values when we cross a line of the form

$$q_i x + (1 - q_i)y = \beta_j,$$

where $\beta_j$ is where two blocks meet.

The nonzero regions of $g(x, y)$ are convex polygons bounded by these lines.
An example for \( \{0, \frac{1}{2}, 1\} \)

A plot of the function \( g(x, y) \) is shown below (red and blue indicate where red and blue three term APs are located).
Using the approximate block structure for \( \{0, \frac{1}{3}, 1\} \) we get:

\[
\gamma = \frac{56816777}{750000000} = 0.07575570 \ldots
\]
Observation

If our block sizes are locally minimal with respect to calculating $\gamma$, then a small perturbation in one of the $\beta_j$ terms cannot decrease $\gamma$.

\[
\left( \text{amount of change in red under } \epsilon \text{ perturbation of } \beta_j \right) + \left( \text{amount of change in blue under } \epsilon \text{ perturbation of } \beta_j \right) = 0
\]
\[ q_{i'}x + (1 - q_{i'})y = \beta_i \]
\[ q_{j'}x + (1 - q_{j'})y = \beta_j + \epsilon \]
\[ q_{j'}x + (1 - q_{j'})y = \beta_j \]
\[ q_{k'}x + (1 - q_{k'})y = \beta_k \]

\[ \Delta \text{Area} \approx \frac{\Delta x}{1 - q_{j'}} \epsilon \]
\[ \approx \left( \frac{1}{q_{j'} - q_{i'}} \beta_i + \frac{1}{1 - q_{j'}} \left( \frac{1 - q_{k'}}{q_{j'} - q_{k'}} + \frac{1 - q_{i'}}{q_{i'} - q_{j'}} \right) \beta_j + \frac{1}{q_{k'} - q_{j'}} \beta_k \right) \epsilon. \]
For a constellation pattern with $k$ points this sets up $k$ linear equations in $k$ unknowns that we can now solve. This gives our locally optimal block structure.

For $\{0, \frac{1}{3}, 1\}$ solving this system gives that the locally optimal block structure has relative block sizes of


Calculating for this pattern we have

$$\gamma = \frac{16040191}{211735908} = 0.075755648 \ldots.$$  

(Note that randomly we would expect $\frac{1}{12} = 0.0833333 \ldots$)
We can now repeat this same procedure for different constellation patterns. Below is a chart recording the outcome of three point constellations \( \{0, q, 1\} \) where \( q \) is given in the \( x \)-axis and on the \( y \)-axis we record how well the pattern that was discovered did when compared to random coloring.
Random is never best

Given $2a < b$, let $0 < \epsilon < 1 + \frac{a}{b} - \frac{a}{b} \left\lfloor \frac{b}{a} \right\rfloor$. Then for the constellation pattern $\{0, \frac{a}{b}, 1\}$ the coloring found by scaling the block pattern

$$(1 - \epsilon) - (1 + \epsilon) - 1 - 1 - \ldots - 1$$

has

$$\gamma = \begin{cases} 
\frac{1}{4b} + \frac{(2a - a \left\lfloor b/a \right\rfloor)}{8ab^2(b-a)} \epsilon + O(\epsilon^2) & \text{if } \left\lfloor b/a \right\rfloor \text{ odd,} \\
\frac{1}{4b} + \frac{(a - 2b + a \left\lfloor b/a \right\rfloor)}{8ab^2(b-a)} \epsilon + O(\epsilon^2) & \text{if } \left\lfloor b/a \right\rfloor \text{ even.}
\end{cases}$$

In particular, it beats random!
This appears to be continuous. (Expected since if we fix our block pattern and perturb $q$ by a small amount then we will only slightly perturb our percentage of random by a small amount.)

The point at $q = \frac{2}{5}$ (and by symmetry $q = \frac{3}{5}$) seems to be out of place!
Local perturbation run for \(\{0, \frac{2}{5}, 1\}\)
Alternating blocks for \( \{0, \frac{2}{5}, 1\} \)

Looking at the outcome of these runs we see

\[
\cdots RBRBRBRRBRBRBRBRBRBRBRBRBRBR \cdots
\]

If we look at every other term then we see blocks, the extra \( R \) has the effect of changing the parity of the location of the \( Rs \) and \( Bs \) between two blocks. We call these alternating blocks.
Blowing up an alternating block pattern gives us

$$\gamma = \frac{1}{20} + \frac{1}{8} \int_0^1 \int_{3y/5}^{(2+3y)/5} f(x)f(y) \, dx \, dy.$$  

This is the same as minimizing the amount of red inside a parallelogram as shown below.
Arithmetic progressions

3AP: Using 12 blocks we have 85.4% of random with $\gamma = \frac{117}{2192}$.

4AP: Using 36 blocks we have 82.6% of random with

$$\gamma = \frac{1793962930221810091247020524013365938030467437975}{104177418768222598213753754515890676996254443021344}.$$ 

5AP: Using 117 blocks we have 73.2% of random with

$$\gamma = \frac{32168(\ldots225 \text{ digits}\ldots)87809}{5624321(\ldots225 \text{ digits}\ldots)51792}.$$
We have been able to use our technique to find many coloring that beat random. But we have not proven that any of these colorings are “best”. For 3APs the best known lower bound is due to Parrilo-Robertson-Saracino

\[
\frac{1675}{32768} = 0.0511169 \ldots \leq \gamma \leq \frac{117}{2192} = 0.0533759 \ldots.
\]

Related to the following geometric problem:
Subdivide \([0, 1]\) into finitely many blocks on the \(x\)-axis, use the same subdivision for \([0, 1]\) on the \(y\)-axis and create a checkerboard pattern (blue in the lower-left corner). Maximize the amount of red inside the triangle with vertices at \((0, 0)\), \((0, 1)\) and \((1, 1/2)\).

Best known pattern is a scaled version of 28-6-28-37-59-116.
Open problems

- Why is symmetry (and in particular anti-symmetry) so common?
- Why does the constellation pattern \(\{0, \frac{2}{5}, 1\}\) go to alternating blocks? Which other constellations should also have that behavior?
- Show that any constellation pattern with 4 or more points has a coloring that beats random.
- Find a coloring that ties random for \(k\)-term arithmetic progressions.
- What happens when we allow for more than two colors?