Symmetric 0-1 matrices with inverses having two distinct values and constant diagonal

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Abstract

This paper examines 0-1 symmetric matrices for which the inverse of the matrix has all entries of the form ±α for some α and constant diagonal. Several constructions of such matrices are given as well as a strong connection to matrices that are invertible and have all principal submatrices of order n − 2 singular.

Keywords: equimodular; constant diagonal; enhanced principal rank

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1. Introduction

In this paper we will be looking at symmetric 0-1 matrices. Such matrices can be interpreted as the adjacency matrix of a graph where a diagonal entry equal to 1 is a loop. For every matrix A there is an associated word ℓ₁ℓ₂...ℓₙ, where each ℓᵢ is one of N, S, or A according to the following rule:

ℓᵢ = \begin{align*}
N & \text{ if no } i \times i \text{ principal submatrices have full rank,} \\
S & \text{ if some}, \text{ but not all}, \ i \times i \text{ principal submatrices have full rank,} \\
A & \text{ if all } i \times i \text{ principal submatrices have full rank.}
\end{align*}

This sequence is known as the “enhanced principal rank sequence” which was introduced in [4] where several important properties were established. In particular, we will make use of the following.

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Proposition 1 \((\text{[4]})\). No symmetric real matrix has an enhanced principal rank sequence containing the subwords NNA or NSA.

We are interested in the class of invertible symmetric 0-1 matrices with all principal submatrices of order \(n - 2\) singular. Using Proposition \([4]\), we can express these matrices as symmetric 0-1 matrices whose enhanced principal rank sequence ends NAA, i.e., \(\ell_n = A\) corresponds to invertible, \(\ell_{n-2} = N\) corresponds to all zero principal \(n - 2\) minors, and finally the proposition forces \(\ell_{n-1} = A\). (For brevity in the paper we will sometimes use “matrix ending with NAA”, or when referencing the adjacency matrix of a graph, an “NAA graph”, for shorthand to indicate that the enhanced principal rank sequence ends NAA.)

In related papers the symmetric 0-1 matrices with 0 diagonal were explored looking at principal rank sequences \([1, 3]\). It was discovered that for order 7 matrices only two of these matrices ended with NAA. These two are the adjacency matrix of a cycle, and the adjacency matrix with corresponding graph shown in Figure \([4]\).

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}
\]

Figure 1: An NAA matrix and corresponding graph on 7 vertices

Surprisingly, the inverse of this matrix has the following beautiful form:

\[
\begin{pmatrix}
1 & 1 & -1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & -1 & -1 & -1 & 1 \\
-1 & 1 & 1 & 1 & -1 & 1 & -1 \\
1 & -1 & 1 & 1 & -1 & 1 & -1 \\
-1 & -1 & -1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 1 & 1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & -1 & 1
\end{pmatrix}
\]

Each entry has the same absolute value, with all entries on the diagonal equal. A thorough computer search has verified that for all symmetric 0-1 matrices of order up through 9 ending in NAA, the corresponding inverse has all entries the same in absolute value with constant diagonal. This has led the authors to the following conjecture.

Conjecture. Let \(A\) be a symmetric 0-1 matrix. Then the following are equivalent:

\footnote{One author believes this to be true, another believes this to be false, and the third is undecided.}
(a) The matrix ends \( \text{NAA} \).

(b) Entries of \( A^{-1} \) are of the form \( \pm \alpha \) with constant diagonal.

We note that (b) implies (a) (so the open part of the conjecture is to show why (a) implies (b)). To see why (b) implies (a) first observe that since the matrix \( A \) has an inverse, \( \ell_n = A \), and since it is symmetric with all diagonal entries the same we have that all the \( 2 \times 2 \) minors of \( A^{-1} \) are 0 which implies by Jacobi’s determinantal identity that \( \ell_{n-2} = \text{N} \). So (b) follows from Proposition 1.

**Definition.** We call a real square matrix *equimodular* if all of its entries have the same absolute value.

So we can restate condition (b) as \( A \) is invertible and \( A^{-1} \) is equimodular with constant diagonal entries. (We note the term “equimodular class of matrices” has appeared previously in the literature; our use is not related to that notion.)

The condition in the conjecture that the diagonal of the inverse be constant is important because there exist 0-1 matrices where the inverse matrix is equimodular but the enhanced principal rank sequence does not end in \( \text{NAA} \), the simplest being the following:

\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}^{-1} = \frac{1}{2} \begin{pmatrix}
1 & 1 & -1 \\
1 & -1 & 1 \\
-1 & 1 & 1
\end{pmatrix}
\]

The rest of this paper is organized as follows. In Section 2 we will give some basic results and observations about \( \text{NAA} \) graphs. In Section 3 we show how to take matrices/graphs satisfying the desired property and combine them to form larger graphs and as a consequence form infinite families of vertex transitive \( \text{NAA} \) graphs. In Section 4 we give several local operations on a graph that will preserve the restricted nature of the inverse which can be used to form many examples of these graphs. Finally we give some concluding remarks in Section 5.

2. Basic facts about matrices ending in \( \text{NAA} \)

Having a matrix which ends with \( \text{NAA} \) is a severe constraint on the matrix and the corresponding graph. We have collected several of these results here.

**Proposition 2.** An \( \text{NAA} \) graph without loops is not bipartite.

**Proof.** Simple bipartite graphs on an odd number of vertices are combinatorially singular and so for a bipartite graph either the graph or the graph with one vertex deleted is singular showing the sequence cannot end with \( \text{NAA} \). \( \Box \)

**Proposition 3.** If a symmetric 0-1 matrix ends with \( \text{NAA} \), then the matrix is irreducible, or equivalently, the corresponding graph is connected.
Proof. If the graph is not connected, then deleting a vertex from one component does not give a singular matrix (because $\ell_{n-1} = A$), and so deleting vertices from two different components will also not produce a singular matrix. This contradicts $\ell_{n-2} = N$.

This argument generalizes to establish the following result.

**Proposition 4.** An NAA graph does not contain a bridge.

Proof. Let $e = uv$ be an edge in the graph where the removal of $e$ leaves connected components $G$ and $H$ where $u \in V(G)$ and $v \in V(H)$. Then since the deletion of one vertex leaves the corresponding matrix invertible we can conclude that $G \setminus v$ is invertible and similarly that $H \setminus u$ is invertible. But now deleting both $u$ and $v$ leaves us with an invertible matrix which is a contradiction since deleting two vertices forces the matrix to be singular.

Combining the previous two results together we can conclude that no symmetric 0-1 matrix which has enhanced principal rank sequence ending with NAA has a row sum of 1. This is because such a row would correspond either to an isolated loop (so the graph is not connected) or a leaf (so the graph would have a bridge). In general, if the conjecture holds, then a symmetric 0-1 NAA matrix must have all even row sums. This is because of the following result (our matrices will correspond to the special case where all the entries of $A^{-1}$ scale to ±1).

**Proposition 5.** Suppose $A$ is an invertible 0-1 symmetric matrix of order at least 2 and that $A^{-1} = qB$ where $q$ is rational and $B$ is an integer matrix where all entries are odd. Then all of the row sums of $A$ are even.

Proof. Given a row of $A$ there is some column of $B$ for which the corresponding dot product is 0. If the row had an odd sum, then the dot product would be the sum of an odd number of odd numbers which is odd, and in particular is not 0. Therefore the row sum must be even.

As a side comment, we note that there are many more matrices satisfying the conditions of Proposition 5 than those that have enhanced principal rank sequence ending with NAA (see the comments in the final section). In particular, for matrices of dimensions 3 through 10 the number of matrices as described in Proposition 5 are as follows (determined by exhaustive computer search):

\[2, 2, 10, 32, 266, 3292, 87184, 4280972\]

While it is not possible for an NAA graph to have a bridge we note that it is possible to have a cut vertex. The smallest example of this is shown in Figure 2 and there are infinitely many other examples (see Section 4 for how to construct additional examples).

Finally we conclude this section with an observation about the null space of the principal submatrices of order $n - 2$. 

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Proposition 6. If a symmetric 0-1 matrix ends with \texttt{NAA}, then each principal submatrix of order \(n - 2\) has rank \(n - 3\). In particular, the null space of each such principal submatrix is one dimensional.

Proof. Since \(\ell_{n-2} = \mathbb{N}\), then a principal submatrix of order \(n - 2\) cannot have full rank. On the other hand each such matrix sits inside a principal submatrix of order \(n - 1\) which has full rank and since the rank can only increase by 2 when appending a row and column we must have that the rank is at least \(n - 3\).

3. A family of vertex transitive graphs from tensor products

By Jacobi’s determinantal identity, to prove the conjecture it suffices to show that all principal \(n - 1\) minors of \(A\) are equal. This will force the diagonal of the inverse to be constant. Combined with knowing that all of the \(2 \times 2\) principal minors of the inverse are 0, we can conclude that the off diagonal entries agree with the diagonal entries up to a sign change. In particular, if the graph is vertex transitive, then the effect of deleting one vertex is the same as deleting another. Therefore the conjecture holds for vertex transitive graphs. The goal of this section is to show that this does not hold vacuously.

The simplest vertex transitive \texttt{NAA} graphs are odd cycles. This can be seen by noting that the determinant of such graphs are known to be 2, i.e., \(\ell_n = \mathbb{A}\). On the other hand upon deletion of any two vertices the remaining graph is a bipartite graph of odd order which is combinatorially singular showing that \(\ell_{n-2} = \mathbb{N}\). Proposition 1 then shows that \(\ell_{n-1} = \mathbb{A}\).

The simple example of cycles can be bootstrapped to form much larger examples by use of the tensor product. The tensor product of \(G\) and \(H\) is the graph \(G \times H\) with vertices \(\{(u, v) : u \in V(G), v \in V(H)\}\), and \((u_1, v_1) \sim (u_2, v_2)\) if and only if \(u_1 \sim u_2\) and \(v_1 \sim v_2\). The name comes from the fact that \(A(G \times H) = A(G) \otimes A(H)\), i.e., the matrix tensor product. This product is also sometimes referred to as the Kronecker product or direct product. (We note in some sources that \(G \times H\) denotes the Cartesian product of graphs; we will use \(G \Box H\) when referring to the Cartesian product to differentiate these two.)
Theorem 1. If $G$ and $H$ are graphs with adjacency matrices whose inverses are equimodular with constant diagonal, then the adjacency matrix of $G \times H$ is also an invertible matrix whose inverse is equimodular with constant diagonal.

Proof. This follows by noting that if $(A(G \times H))^{-1} = (A(G))^{-1} \otimes (A(H))^{-1}$ and since $(A(G))^{-1}$ and $(A(H))^{-1}$ are of the desired form, then so is $(A(G \times H))^{-1}$.

We comment here that a similar argument will show that the tensor product of two NAA graphs will be an NAA graph.

Using this theorem we can now construct many more vertex transitive families, i.e., $C_{2k+1} \times C_{2\ell+1}$ as well as larger iterated tensor products of these graphs.

We note that $C_{2k+1} \times C_{2k+1} \cong C_{2k+1} \square C_{2k+1}$, which has the following interesting consequence. The eigenvalues of $C_{2k+1}$ are

$$2, 2\cos \frac{2\pi}{2k+1}, 2\cos \frac{2\pi}{2k+1}, \ldots, 2\cos \frac{2k\pi}{2k+1}, 2\cos \frac{2k\pi}{2k+1}.$$  

Since the eigenvalues of $C_{2k+1} \square C_{2k+1}$ are obtained from all possible sums of pairs of eigenvalues of $C_{2k+1}$ while the eigenvalues of $C_{2k+1} \times C_{2k+1}$ are obtained from all possible products of pairs of these eigenvalues (see [2]), it follows that (*) gives a list of $2k + 1$ numbers whose (multi)-set of sums exactly matches its (multi)-set of products.

We give a proof of the form of the inverse of $A(C_{2k+1} \square C_{2k+1})$ as it is illustrative of the procedure we use in the following section. The primary tool we will use is the following lemma.

Lemma 1. Let $A$ be an invertible matrix of order $n$, let $G$ be the associated graph, let $v$ be a vertex of $G$, let $x \in \mathbb{R}^n$ be a nonzero vector such that $(Ax)(w) = 0$ for all $w \neq v$, and let $c = (Ax)(v)$. Then $(1/c)x$ is the $v$th column of $A^{-1}$, i.e., $(A^{-1})_{v,v} = x(v)/c$.

Proof. First we note that $c \neq 0$ because otherwise the vector $x$ is a nonzero null vector which contradicts that the matrix is invertible. In particular, the vector $A((1/c)x) = e_v$, i.e., the vector which is 1 for $v$ and 0 otherwise, which verifies that $(1/c)x$ is the $v$th column of $A^{-1}$.

Theorem 2. If $A$ is the adjacency matrix of $C_{2k+1} \square C_{2k+1}$, then $A$ ends with NAA.

Proof. Since we have already observed that a matrix whose inverse is equimodular with constant diagonal implies NAA, it suffices to show that the inverse is of the form $\pm \alpha$ with all diagonal entries agreeing. By vertex transitivity it suffices to apply Lemma 1 to a single vertex $v$ in the graph to determine the entries of the inverse matrix.

Let us label the vertices of our graph by $(i,j)$ where $-k \leq i,j \leq k$ and consider the vector $x$ on the vertices of the graph where

$$x = \begin{cases} 1 & \text{if } \min\{|i|,|j|\} \text{ is even}, \\ -1 & \text{if } \min\{|i|,|j|\} \text{ is odd}. \end{cases}$$

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An example of the resulting labeling is shown in Figure 3 when $k = 4$. Note that the edges “wrap around” left to right and top to bottom, though these are not shown in the diagram.

Figure 3: A construction of $x$ when $k = 4$

Except for the vertex $(0,0)$ the sum of the entries of the four vertices adjacent to any vertex is 0. This is because at $(i,j) \neq (0,0)$ in two directions we change $\min\{|i|,|j|\}$ by one and in the other two directions it is constant. At $(0,0)$ the sum of the adjacent vertices is 4. Therefore by Lemma 1 we can conclude that each column of the inverse has entries $\pm \frac{1}{4}$ and that each diagonal of the inverse is $\frac{1}{4}$.

4. Local graph operations

In the previous section we found a family of NAA graphs. In this section we will show that in some cases when we have a graph whose adjacency matrix is invertible with the inverse being equimodular with constant diagonal, then we can apply one of several local operations to construct a larger graph with the same property and in the process construct infinite families.

The first operation will correspond to taking a vertex of degree 2 and replacing it with a longer path.

**Theorem 3.** Let $G$ be a graph whose inverse is equimodular with constant diagonal, and further assume that $G$ has a vertex $v$ of degree 2 with neighbors $u$ and $w$. Let $H$ be the graph obtained by deleting $v$ from $G$ and adding in the path with vertices $v_1, v_2, v_3, v_4,$ and $v_5$ each of degree 2 so that $u \sim v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_5 \sim w$. Then $H$ is also a graph whose inverse is equimodular with constant diagonal.
The construction of this theorem is illustrated in Figure 4.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{The construction corresponding to Theorem 3}
\end{figure}

**Proof.** There exists an $\alpha \neq 0$ such that all entries of the inverse matrix are $\pm \alpha$. By scaling the columns of the inverse by $1/\alpha$ we have $\pm 1$ vectors which satisfy the conditions of Lemma 1. We will show that we can modify these to show that there are $\pm 1$ vectors which satisfy the conditions of Lemma 1 for $H$, which in turn will show that the inverse of $H$ has the correct form. We will do this in several cases.

\textbf{A vertex in $H$ other than $v_1$, $v_2$, $v_3$, $v_4$ or $v_5$:} In this case let $x$ be the $\pm 1$ vector for the corresponding vertex in $G$. Now construct the vector $x'$ for $H$ which agrees with $x$ and where

\[ x'(v_1) = x(v), \quad x'(v_2) = x(w), \quad x'(v_3) = -x(v), \]
\[ x'(v_4) = x(u), \quad x'(v_5) = x(v). \]

Then $x'$ is a $\pm 1$ vector which satisfies Lemma 1 with the same value of $c' = c$ and so this column of the inverse and corresponding diagonal entry will be of the correct form.

\textbf{The vertex $v_1$ or $v_5$:} By symmetry it suffices to work through the case $v_1$. Let $x$ be the $\pm 1$ vector for the vertex $v$ in $G$. Now construct the vector $x'$ for $H$ which agrees with $x$ and where

\[ x'(v_1) = x(v), \quad x'(v_2) = x(w), \quad x'(v_3) = -x(v), \]
\[ x'(v_4) = -x(u), \quad x'(v_5) = x(v). \]

Then $x'$ is a $\pm 1$ vector which satisfies Lemma 1 with the same value of $c' = c$ and $x(v)/c = x'(v_1)/c'$ and so this column of the inverse and corresponding diagonal entry will be of the correct form.

\textbf{The vertex $v_3$:} Let $x$ be the $\pm 1$ vector for the vertex $v$ in $G$. Now construct the vector $x'$ for $H$ which agrees with $x$ and where

\[ x'(v_1) = x(v), \quad x'(v_2) = -x(u), \quad x'(v_3) = -x(v). \]
\[ x'(v_4) = -x(w), \quad x'(v_5) = x(v). \]

Then \( x' \) is a \( \pm 1 \) vector which satisfies Lemma 1 with the value of \( c' = -c \) and \( x(v)/c = x'(v_3)/c' \) and so this column of the inverse and corresponding diagonal entry will be of the correct form.

The vertex \( v_2 \) or \( v_4 \): By symmetry it suffices to work through the case \( v_2 \).

Let \( x \) be the \( \pm 1 \) vector for the vertex \( u \) in \( G \), and let \( y = (\sum_{t \sim w} x(t)) - x(v) \).

Now construct the vector \( x' \) for \( H \) which agrees with \( x \) and where
\[
x'(v_1) = -y, \quad x'(v_2) = -x(u), \quad x'(v_3) = -x(v),
\]
\[
x'(v_4) = -x(w), \quad x'(v_5) = x(v).
\]

The vector \( x' \) satisfies Lemma 1 for \( H \) at \( v_2 \) with the value of \( c' = -c \) and \( x(u)/c = x'(v_2)/c' \) and so this column of the inverse and corresponding diagonal entry will be of the correct form as long as \( y \in \{-1, 1\} \).

To show that \( y \in \{-1, 1\} \) we now give an alternate construction. Let \( y \) be the \( \pm 1 \) vector for the vertex \( w \) in \( G \), and let \( z = (\sum_{t \sim w} y(t)) - y(v) \). Now construct the vector \( y' \) for \( H \) which agrees with \( y \) in \( G \) and where
\[
y'(v_1) = y(v), \quad y'(v_2) = y(w), \quad y'(v_3) = z,
\]
\[
y'(v_4) = -y(w), \quad y'(v_5) = -z.
\]

The vector \( y' \) again satisfies Lemma 1 for \( H \) at \( v_2 \), and so must agree up to scaling with \( x' \), and since \( x(u) = \pm y(u) \) it differs at most in sign. Therefore \( y \in \{-1, 1\} \) as desired, finishing this case and the proof.

As an example of an infinite family we can construct, consider the graph and associated matrix shown in Figure 5 which has an inverse that is equimodular with constant diagonal.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0
\end{pmatrix}
\]

Figure 5: An \texttt{NAAA} matrix and corresponding graph on 9 vertices

Using this graph along with the one given in Figure 1, we obtain the following result where the family is constructed by simply repeatedly applying the operation in Theorem 3 to vertices of degree 2 on the base graphs.

**Corollary 1.** Let \( H \) be a graph formed by taking \( C_{2k+1} \), \( C_{4p} \), and \( C_{4q} \) for \( k,p,q \geq 1 \) and gluing the cycles together along a shared common edge (i.e., as...
illustrated in Figure 5 for \( k = 2 \) and \( p = q = 1 \). Then the inverse of the adjacency matrix is equimodular with constant diagonal.

The second local operation will correspond to adding in a square.

**Theorem 4.** Let \( G \) be a graph whose inverse is equimodular with constant diagonal, and further assume that \( G \) has vertices \( v_1, v_2, u_1, u_2 \) with \( v_1 \sim u_1 \sim u_2 \sim v_2 \), the degrees of \( u_1 \) and \( u_2 \) are both 2, and that \( v_1 \) is not adjacent to \( v_2 \). Let \( H \) be the graph obtained by adding the vertices \( v_3 \) and \( v_4 \) and edges so that \( v_1 \sim v_2 \sim v_3 \sim v_4 \sim v_1 \) and \( v_3 \) and \( v_4 \) have degree 2. Then \( H \) is also a graph whose inverse is equimodular with constant diagonal.

The construction of this theorem is illustrated in Figure 6.

**Proof.** There exists an \( \alpha \neq 0 \) such that all entries of the inverse matrix are \( \pm \alpha \). By scaling the columns of the inverse by \( 1/\alpha \) we have \( \pm 1 \) vectors which satisfy the conditions of Lemma 1. We now show how to expand these to form the corresponding vectors for \( H \).

Suppose that we consider a vertex in \( H \) other than \( v_3 \) or \( v_4 \). Then we start by using the corresponding vector \( x \) for that vertex for \( G \) and construct a new vector \( x' \) for \( H \) by defining

\[
x'(v_3) = -x(v_1), \quad x'(v_4) = -x(v_2).
\]

The vector \( x' \) is a \( \pm 1 \) vector which satisfies Lemma 1 for \( H \) with the same value of \( c \) as \( x \) in \( G \), and so the column of the inverse and corresponding diagonal entry will be the same.

On the other hand, by symmetry we can map \( v_3 \) to \( u_2 \) and \( v_4 \) to \( u_1 \) and therefore since we already have established that \( u_1 \) and \( u_2 \) work as needed, then so also will \( v_3 \) and \( v_4 \), concluding the proof.

An example of a graph formed using this construction is shown in Figure 7 which is constructed by starting with \( C_{15} \) and repeatedly applying Theorem 4.
The third and final operation will involve “gluing” on a particular graph. Namely, take a cycle $C_n$ and then add a pendant leaf to each vertex of this cycle forming a graph known as the $n$-sun (this is also sometimes referred to in the literature as the $n$-sunlet, or the pin-cycle). Now we glue the $n$-sun on by connecting every vertex of the $n$-sun to some fixed vertex $v$ in our graph. This method was used to construct the graph in Figure 2 where we glued the 3-sun into the cycle $C_5$.

**Theorem 5.** Let $G$ be a graph whose inverse has entries of the form $\pm \frac{1}{2}$ with all diagonal terms equal to $\frac{1}{2}$, let $v$ be any vertex in $G$, and let $n \geq 3$. Let $H$ be the graph obtained by taking the disjoint union of $G$ and an $n$-sun and then connecting every vertex in the $n$-sun to $v$. Then $H$ is a graph whose inverse has all entries of the form $\pm \frac{1}{2}$ with all diagonal terms equal to $\frac{1}{2}$.

Note that this theorem differs from the two previous ones in that we restrict the choice of $\alpha$ to be $\frac{1}{2}$, hence this cannot be applied to all of the graphs in our class. However, for most small graphs we have $\alpha = \frac{1}{2}$ and further $\alpha = \frac{1}{2}$ for $C_{4k+1}$. Also we can iteratively apply this theorem and thus form graphs with arbitrarily many cut vertices so that the corresponding adjacency matrix has an inverse which is equimodular and with constant diagonal.

**Proof.** By assumption, all entries of the inverse matrix are $\pm \frac{1}{2}$. By scaling the columns of the inverse by 2 we have $\pm 1$ vectors which satisfy the conditions of Lemma[1]. We now show how to expand these to form the corresponding vectors for $H$. We will do this in several cases.

For convenience we will label the vertices in the $n$-sun as $u_1, u_2, \ldots, u_k$, $w_1, w_2, \ldots, w_k$, where the $w_i$ are the vertices of the cycle listed in cyclic order and the $u_i$ are the pendants satisfying $u_i \sim w_j$.

A vertex in $H$ that was originally in $G$: In this case let $x$ be the $\pm 1$ vector for the corresponding vertex in $G$. Now construct the vector $x'$ for $H$ which agrees with $x$ and for $1 \leq i \leq k$

$$x'(u_i) = x(v), \quad x'(w_i) = -x(v).$$
This vector is a $\pm 1$ vector which satisfies Lemma 1 with the same value of $c$ and so this column of the inverse and corresponding diagonal entry will be of the correct form.

A vertex among the $u_i$: By symmetry we may without loss of generality assume it is $u_1$. Let $x$ be the $\pm 1$ vector for the vertex $v$ in $G$ which is 1 for $v$. Note by our assumption on the diagonal entry that the sum of the weights on the the neighbors of $v$ will be 2. Now construct the vector $x'$ for $H$ which agrees with $x$ on the vertices of $G$ and where

$$x'(u_i) = \begin{cases} -1 & i = 2 \text{ or } k \\ 1 & \text{otherwise} \end{cases}$$

$$x'(w_i) = \begin{cases} 1 & i = 1 \\ -1 & \text{otherwise} \end{cases}$$

The vector $x'$ is a $\pm 1$ vector. There are now two more vertices labeled $-1$ than 1 among the new vertices which corrects the imbalance at $v$, and a simple check verifies that for every vertex other than $u_1$ the sum of the weights of the neighbors is 0, while for $u_1$ the weights of the neighbors $v$, $w_1$ sum to 2. Therefore by Lemma 1 this column of the inverse and corresponding diagonal entry will be of the correct form.

A vertex among the $w_i$: By symmetry we may without loss of generality assume it is $w_1$. Let $x$ be the $\pm 1$ vector for the vertex $v$ in $G$ which is 1 for $v$, which again has an imbalance of 2. Now construct the vector $x'$ for $H$ which agrees with $x$ on the vertices of $G$ and where

$$x'(u_i) = \begin{cases} -1 & i = 1 \\ 1 & \text{otherwise} \end{cases}$$

$$x'(w_i) = -1$$

The vector $x'$ is a $\pm 1$ vector. There are now two more vertices labeled $-1$ than 1 among the new vertices which corrects the imbalance at $v$, and a simple check verifies that for every vertex other than $w_1$, the sum of the weights of the neighbors is 0, while for $w_1$ the weights of the neighbors $v$, $u_1$, $w_2$ and $w_k$ sum to $-2$. Therefore the diagonal entry will be $x'(w_1)/(-2) = \frac{1}{2}$ as needed. So these columns will also be of the correct form, concluding the proof. □

5. Concluding remarks

This paper has been a cursory introduction to symmetric 0-1 matrices for which the inverse has the special form of having all entries $\pm \alpha$ with a constant diagonal. We have computational and theoretical connections between such matrices and matrices ending with $NAA$. We note that because of the strict requirements placed on these matrices that they are rare. We list here all such matrices (up to permutation similarity) through order 9.

- For order 3 the only matrix is the adjacency matrix for $C_3$. 
- For order 5 the only matrix is the adjacency matrix for $C_5$.
- For order 7 the only matrices are the adjacency matrix for $C_7$ and the matrix given in Figure 1.
- For order 9 the only matrices are the adjacency matrices for $C_9$, $C_3 \times C_3$, the matrix given in Figure 5 and the following eight additional matrices:

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}
There are no matrices in our family for orders 2, 4, 6, or 8. If the conjecture holds, then the only matrix for \( n = 10 \) is the following.

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]

Similarly, if the conjecture holds, then for order 11 matrices there are 102 ending in \text{NAA}, all of which are adjacency matrices of graphs without loops\(^2\).

While establishing the conjecture would of course be the most desirable goal, it would be interesting to see what structural properties the enhanced principal rank sequence ending with \text{NAA} can force upon a matrix.

We comment here that all of the matrices found so far have \( \alpha = \frac{1}{2^k} \), and further by taking iterated tensor products we can realize matrices in this family where \( k \) is arbitrarily large. One open problem is whether the matrices can have any other value of \( \alpha \), e.g., \( \alpha = \frac{1}{6} \).

Further, we have given only some very rudimentary constructions and operations. These do not explain most of the graphs found by computation, including some with interesting symmetry and structure (see Figure 8 for an example).

![Figure 8: One of many interesting unexplained NAA graphs](https://sage.math.iastate.edu/home/pub/25/)

Even partial results for these graphs and matrices would be interesting, and we look forward to seeing more progress in understanding the structure of 0-1 symmetric matrices with enhanced principal rank sequence ending in \text{NAA}.

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\(^2\)Available online at: [https://sage.math.iastate.edu/home/pub/25/](https://sage.math.iastate.edu/home/pub/25/)
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