Epidemic outbreaks in networks with equitable or almost-equitable partitions

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Epidemic outbreaks

- Model an outbreak on a graph
- Vertices are people or computers
- Edges are connections

- Vertex can either be infected or healthy (susceptible to infection)
Epidemic outbreaks

Assumptions

- Stable social communities (no migration)
- Continuous-time Markovian SIS model

What we want to understand

- For a finite system, always converge to an all healthy state
- **Epidemic threshold** where the time to convergence grows exponentially or decreases exponentially
Effective Spreading Rate

- At each time $t$, each vertex has a probability of getting infected $p_i(t)$
- The rate at which the infection spreads is $\beta$
- The rate at which a vertex recovers is $\delta$
- Effective spreading rate $\tau = \frac{\beta}{\delta}$
Epidemic Threshold

- Epidemic threshold $\tau_c$ is critical value of $\tau$
- If $\tau > \tau_c$, then unrealistic long time before the healthy state is reached
- If $\tau < \tau_c$, then infection vanishes in exponentially fast time
- For a given graph, we know that $\frac{1}{\lambda_1(A)}$ is a lower bound for $\tau_c$
Equitable partitions

- Quotient matrix: same spectral radius as adjacency
- Allows us find estimate for threshold more efficiently

- Partition vertices of $G$ into $V_1, \ldots, V_n$ where each $V_i$ induces a regular subgraph and the vertices in $V_i$ have the same number of neighbors in each $V_j$
Equitable partitions, cont.

\[ Q = \begin{pmatrix}
\deg V_1 & \cdots & \frac{e(V_1,V_n)}{\sqrt{|V_1||V_n|}} \\
\vdots & \ddots & \vdots \\
\frac{e(V_1,V_n)}{\sqrt{|V_1||V_n|}} & \cdots & \deg V_n
\end{pmatrix} = D + B \]

\[ \lambda_1(Q) \leq \lambda_1(D) + \lambda_1(B) = \max (\deg V_i) + \lambda_1(B) \]

Since \( \lambda_1(A) = \lambda_1(Q) \), it follows \( \tau_c \geq \min_i \frac{1}{\deg(V_i) + \lambda_1(B)} \)
Almost-equitable partitions

- We had an equitable partition and then added edges inside the sets.
- We had an equitable partition, then removed edges inside the set.

- Note that removing and adding edges only happens within the partitions
Added Edges

- Let $G$ be a graph that has an equitable partition
- Let $G'$ be a copy of $G$ with added edges.
- $A' = A + R$
- $\lambda_1(A') \leq \lambda_1(A) + \lambda_1(R)$
Investigation of $\lambda_1(R)$

- Let $G_i$ be the graph induced by the edges added within each partition. Let the number of edges of each $G_i$ be $e_i$. Let the number of vertices of each $V_i$ be $k_i$.

- Claim: $\lambda_1(R) \leq \max_i \left( \min \left( \sqrt{\frac{2e_i(k_i-1)}{k_i}}, \Delta(G_i) \right) \right)$
Proof of Claim: $\lambda_1(R) \leq \max_i \left( \min \left( \sqrt[2e_i(k_i-1)]{\frac{k_i}{k_i}}, \Delta(G_i) \right) \right)$

- From a result that bounds zeros of a polynomial using the Holder inequality, we know that all the eigenvalues of $R$ are in the interval
  
  $$-\sqrt[2e_i(k_i-1)]{\frac{k_i}{k_i}} \leq \lambda(R_i) \leq \sqrt[2e_i(k_i-1)]{\frac{k_i}{k_i}}$$

- By Gerschgorin’s Theorem, we know that $\lambda_1 \leq \Delta(G)$
Epidemic Threshold of Almost-Equitable Partitions

\[ \tau_c \geq \frac{1}{\max_i(\text{deg}_G(V_i)) + \lambda_1(B) + \max_i \left( \min \left( \sqrt{\frac{2e_i(k_i - 1)}{k_i}}, \Delta(G_i) \right) \right)} \]
Removing Edges

- Let $G$ be a graph that has an equitable partition.
- Let $G'$ be a copy of $G$ with removed edges.
- $R^+ A' = A$
- $\lambda_1(A') \leq \lambda_1(A)$

So, $\tau_c \geq \frac{1}{\max_i (\deg_G(V_i)) + \lambda_1(B)}$
Comparison of two bounds

**Remove edges**

\[ \tau_c \geq \frac{1}{\max_i (\deg_G(V_i)) + \lambda_1(B)} \]

**Add edges**

\[ \tau_c \geq \frac{1}{\max_i (\deg_G(V_i)) + \lambda_1(B) + \max_i \left( \min \left( \sqrt{\frac{2e_i(k_i - 1)}{k_i}}, \Delta(G_i) \right) \right)} \]
Summary/Good exam questions?

- We can estimate the epidemic threshold of a network using the largest eigenvalue
- We can estimate the largest eigenvalue in equitable partitions
- We can estimate the largest eigenvalue in almost-equitable partitions
Work Cited