Erdős–Ko–Rado for Permutations

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Spectral Graph Theory
Math 595

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A new proof of the Erdős–Ko–Rado theorem for intersecting families of permutations

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Outline:

Definitions
Theorem
Proof
Theorem: (Erdős, Ko, Rado, 1961)

Let $k, n$ be positive integers with $n > 2k$. If $A$ is a family of $k$-subsets of \{1, 2, \ldots, n\} such that any two sets from $A$ have non-trivial intersection, then $|A| \leq \binom{n-1}{k-1}$. Moreover, we have equality if and only if $A$ is the collection of all $k$-subsets containing a fixed $i \in \{1, 2, \ldots, n\}$.

Proof.

Exercise.
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Proof.

Exercise.
$n = 4, k = 2$

Illustration courtesy of Wikipedia.
For $S_n$ the symmetric group on $\{1, 2, \ldots, n\}$, permutations $\pi, \sigma \in S_n$ are said to be *intersecting* if $\pi(i) = \sigma(i)$ for some $i \in \{1, 2, \ldots, n\}$.

There is an analogous “good guess” for maximum families of intersecting permutations:

$$S_{i,j} = \{ \pi \in S_n : \pi(i) = j \}, \quad i, j \in \{1, 2, \ldots, n\}$$

We may think of these sets as the cosets of the stabilizer of point $i$ with respect to the action of $S_n$ on $\{1, 2, \ldots, n\}$.

e.g. $S_{n,n}$ is the set of all permutations fixing $n$. 
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Theorem: (Cameron and Ku, Larose and Malvenuto, Godsil and Meagher)

Let $n \geq 2$. If $S \subseteq S_n$ is an intersecting family of permutations, then:

(a) $|S| \leq (n - 1)!$
(b) if $|S| = (n - 1)!$, then $S$ is a coset of a stabilizer of a point.

Proof.

Various techniques.

One of them the topic of this paper!
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To prove part (a) of the theorem, we employ what is called the *clique-coclique bound*.

This was originally proven by Philippe Delsarte in 1973 using association schemes.

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Theorem: (Cameron, Ku, 2003)

Let $G$ be a vertex-transitive graph of order $n$. Suppose that $T$ is a subset of $V(G)$ and that the largest clique contained in $T$ has size $\frac{|T|}{m}$. Then any clique $S$ in $G$ satisfies $|S| \leq \frac{n}{m}$. Equality implies that $|S \cap T| = \frac{|T|}{m}$. 
Proof.

Count pairs \((v, \varphi)\) where \(v \in S\), \(\varphi \in \text{Aut}(G)\), and \(\varphi(v) \in T\). For each \(w \in T\) there are \(\frac{|\text{Aut}(G)|}{n}\) choices of \(\varphi\) so that \(\varphi(v) = w\), so the number of pairs is \(|S| \cdot |T| \cdot \frac{|\text{Aut}(G)|}{n}\). For any \(\varphi \in \text{Aut}(G)\), however, we also have \(|\varphi(S) \cap T| \leq \frac{|T|}{m}\) since \(\varphi(S) \cap T\) is a clique in \(T\); thus the number of pairs is at most \(|\text{Aut}(G)| \cdot \frac{|T|}{m}\). So

\[
|S| \cdot |T| \cdot \frac{|\text{Aut}(G)|}{n} \leq |\text{Aut}(G)| \cdot \frac{|T|}{m}
\]

and

\[
|S| \leq \frac{n}{m}.
\]

In the case of equality, \(|\varphi(S) \cap T| = \frac{|T|}{m}\) for all \(\varphi \in \text{Aut}(G)\), so choose \(\varphi = \text{id}\).
Corollary: (The clique-coclique bound)

Let $C$ be a clique and $A$ a coclique in a vertex-transitive graph of order $n$. Then $|C| \cdot |A| \leq n$. Equality implies that $|C \cap A| = 1$. 
The derangement graph $P(n)$

For positive integer $n$, we define $P(n)$ to be the graph whose vertex set is $S_n$ and with edge $\pi\sigma$, for $\pi, \sigma \in S_n$, if and only if $\pi$ and $\sigma$ are non-intersecting, that is $\pi(i) \neq \sigma(i)$ for all $i \in \{1, 2, \ldots, n\}$.

$P(n)$ is called the derangement graph because it is a Cayley graph generated by the complete set of derangements of $\{1, 2, \ldots, n\}$: all permutations with no fixed points.

Note: Cayley graphs are vertex-transitive!

A small example:
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Theorem: clique number of $P(n)$

$\omega(P(n))$, the size of a maximum clique in $P(n)$, is $n$.

Proof.

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Theorem: independence number of $P(n)$

$\alpha(P(n))$, the size of a maximum independent set in $P(n)$, is $(n - 1)!$.

Proof.

Since $P(n)$ is vertex-transitive, the clique-coclique bound holds:

$$\alpha(P(n)) \cdot \omega(P(n)) \leq |P(n)|$$

and so

$$\alpha(P(n)) \leq \frac{n!}{n} = (n - 1)!.$$
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This completes the proof of part (a) of Erdős–Ko–Rado for intersecting families of permutations.

Proving part (b) (that for intersecting family of permutations $S \subseteq S_n$ if $|S| = (n - 1)!$, then $S$ is a coset of a stabilizer of a point) is not easy:

Godsil and Meagher make use of association schemes, $[n - 1, 1]$-modules, character theory and several pages of hardcore linear algebra.
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Some spectral results on $P(n)$:
Lemma: (Godsil and Meagher)

For all positive integers $n$, $D_n$ and $-\frac{D_n}{n-1}$ are eigenvalues of $P(n)$, where $D_n$ is the derangement number of $n$.

Proof.

Take independent set $S_{n,n}$ as previously defined and note that $\{S_{n,n}, V(P(n)) - S_{n,n}\}$ is the orbit partition of $S_1 \times S_{n-1}$ acting on $P(n)$, and is thus an equitable partition. The adjacency matrix with respect to this partition is

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\begin{pmatrix}
0 & D_n \\
\frac{D_n}{n-1} & D_n - \frac{D_n}{n-1}
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which has eigenvalues $D_n$ and $-\frac{D_n}{n-1}$, and so $P(n)$ does as well.
Lemma: (Godsil and Meagher)

For all positive integers \( n \), \( D_n \) and \( -\frac{D_n}{n-1} \) are eigenvalues of \( P(n) \), where \( D_n \) is the derangement number of \( n \).

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which has eigenvalues \( D_n \) and \( -\frac{D_n}{n-1} \), and so \( P(n) \) does as well.
Moreover, both of these values are integers: as some may recall from Math 606 we have recurrence relation

\[ D_n = (n - 1)(D_{n-1} + D_{n-2}) \]

\[ \frac{D_n}{n-1} = -(D_{n-1} + D_{n-2}) \]

\( D_n \) is the regularity of \( P(n) \) and is thus its largest eigenvalue, and in 2007 Renteln showed that \( -\frac{D_n}{n-1} \) is its smallest.
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Thank you!