Basic information about the test

The test is 8:15pm-9:45pm on Thursday, February 23. You are encouraged to arrive 10-15 minutes early to settle. Note that the location might not be your classroom, so make sure to know where you are supposed to go take the test. Bring an extra pen/pencil, dress comfortably, avoid 64 oz. drinks before the exam (or sit in the aisle seat if you think you might need to use the restroom). If needed, bring cough drops and/or tissues.

Calculators, including scientific and graphing calculators, are allowed. At most you need a calculator for arithmetic. The exam has one part, with seven problems. Each problem is worth seven points. There will be a blank page on the back; no other scratch paper is allowed.

Test taking strategies and advice

- If you are not making progress on a problem, move to another problem and come back later.
- Check your work often. No problem requires "crazy" computations. If the problem is becoming crazy stop and see if you made a mistake. (Most mistakes are simple copy or arithmetic mistakes.)
- Prepare by practicing problems and reviewing ideas. Work on hard problems before looking at solutions! The more you struggle with a problem the more you will be prepared to answer similar problems in the future.
- Get a good night’s rest before the exam. Falling asleep during the exam is rarely successful.
- You have seen everything that you will be tested on. Don’t panic, breathe in, breathe out.

Ten basic integrals

\[ \int \frac{du}{u} = \ln|u| + C \]
\[ \int u^n \, du = \frac{u^{n+1}}{n+1} + C \]
\[ e^u \, du = e^u + C \]
\[ \int \frac{du}{1 + u^2} = \arctan u + C \]
\[ \cos u \, du = \sin u + C \]
\[ \sin u \, du = -\cos u + C \]
\[ \sec^2 u \, du = \tan u + C \]
\[ \sec u \tan u \, du = \sec u + C \]
\[ \int \tan u \, du = \ln|\sec u| + C \]
\[ \int \sec u \, du = \ln|\sec u + \tan u| + C \]

(Should also know the corresponding derivatives.)

\[ \int f(g(x))g'(x) \, dx = \int f(u) \, du \quad \text{by } u = g(x) \]

Finding volumes and volumes of revolution

Break into “slices”, find volume of slices, add up.

\[ V = \int_{a}^{b} A(x) \, dx \quad A(x) = \text{area of cross section} \]

For volumes of revolution we have washer method

\[ V = \pi \int_{a}^{b} \left( \left( \text{outer radius} \right)^2 - \left( \text{inner radius} \right)^2 \right) \, dx \]

and shell method

\[ V = 2\pi \int_{a}^{b} (\text{radius})(\text{height}) \, dx \]

\[ = 2\pi \int_{a}^{b} x(f(x) - g(x)) \, dx \]

Arc length and surface area

Find arc length by breaking curves into small parts.

\[ L = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx \]

Find surface area by “rotating” the arc length.

\[ SA = 2\pi \int_{a}^{b} (\text{radius})\sqrt{1 + (f'(x))^2} \, dx \]

around the x-axis the radius is \( f(x) \);
around the y-axis the radius is \( x \).
Work
Work is \( \text{(Force)} \times \text{(Distance)} \), when force \( F(x) \) varies,
\[
\text{Work} = \int_a^b F(x) \, dx.
\]
For pumping out of a container we have
\[
\text{Work} = \int_a^b (\text{density})(\text{distance})(\text{area of slice}) \, dy.
\]
When fluid pushes against a plate we have
\[
\text{Force} = \int_a^b (\text{density})(\text{depth})(\text{width}) \, dy.
\]

Mass and center of mass
Given a density function \( \delta(x) \) the mass of a rod is
\[
M = \int_a^b \delta(x) \, dx,
\]
and the moment with the origin is
\[
M_0 = \int_a^b x \delta(x) \, dx.
\]
From these we can find the “weighted average” of \( x \), denoted \( \bar{x} = \frac{M_0}{M} \).
When our region is two dimensional, we similarly find mass, and now we have two different moments
(with respect to the two axes). Using this we can find the center of mass \((\bar{x}, \bar{y})\). If the region and the density have symmetry across a line then the center of mass lies on that line (sometimes simpler).
\[
\begin{align*}
M &= \int_a^b \delta(x)(f(x)-g(x)) \, dx \\
M_y &= \int_a^b \delta(x)x(f(x)-g(x)) \, dx \\
M_x &= \int_a^b \frac{1}{2} \delta(x)((f(x))^2-(g(x))^2) \, dx \\
(\bar{x}, \bar{y}) &= \left( \frac{M_y}{M}, \frac{M_x}{M} \right)
\end{align*}
\]

Integration by parts
\[
\int u \, dv = uv - \int v \, du
\]
Integration by parts comes from the product rule, so we look for integral of one part and integrate one part; hopefully making it easier! Might have to do integration by part several times; and might also need to rearrange terms (after some steps we return to where we started, e.g., sec^3 \( \theta \)). If trying to integrate a function we don’t know we can add a “1” and do by parts, i.e., \( \int 1 \cdot f(x) \, dx = xf(x) - \int xf’(x) \, dx \).

Trigonometric integrals
When dealing with integrals of trigonometric functions, try to get into one of the following forms
\[
\begin{align*}
\sin \theta &- \cos \theta \\
\tan \theta &- \sec \theta \\
\sec \theta &- \tan \theta
\end{align*}
\]
If we can’t get it into one of these, we either use power reduction formulas on \( \sin^2 \theta \) and \( \cos^2 \theta \); or we write everything in terms of sec \( k(x) \) where \( k \) is odd and be miserable. (Only use \( \sin^2 \theta + \cos^2 \theta = 1 \) or \( \tan^2 \theta + 1 = \sec^2 \theta \) in the numerator, if possible.)

Trigonometric substitution
Square roots are hard, but common. To integrate when square roots are involved we use trigonometry as follows:
\[
\begin{align*}
\sqrt{1-u^2} &- \text{ use } u = \sin \theta \quad du = \cos \theta \, d\theta \\
\sqrt{1+u^2} &- \text{ use } u = \tan \theta \quad du = \sec^2 \theta \, d\theta \\
\sqrt{u^2-1} &- \text{ use } u = \sec \theta \quad du = \sec \theta \tan \theta \, d\theta
\end{align*}
\]
After making the appropriate substitution the square root term will go away, and then we have a trigonometric integral which we can (hopefully) do. After finishing remember to go back to your original variable, drawing a triangle might be helpful.

Partial fractions
To integrate a \( (\text{polynomial})/(\text{polynomial}) \) we (1) do long division if needed; (2) factor the denominator; (3) use partial fractions to “pull apart” our integral into basic pieces which have the form,
\[
\frac{A}{(x-a)^k} \quad \text{or} \quad \frac{Bx+C}{(x-b)^2+c^2}^\ell 
\]
when solving for constants first try good \( x \) and then go to coefficients; (4) finish.

Improper integrals
An improper integral is one which involves \( \infty \), either as one or more of the bounds, or a vertical asymptote. To find we use limits of proper integrals, e.g., \( \int_a^b f(x) \, dx = \lim_{b \to 1-} \int_0^b f(x) \, dx \). Note we break our integral into parts, one part for each “\( \infty \)” if any part does not converge then the whole thing will diverge.
Can use L’Hôpital’s rule when taking limits:
\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}
\]
Note, often easier to first find the indefinite integral and avoid dealing with bounds, until the end.