Points and distance

One way to describe our position in three dimensional space is using Cartesian coordinates \((x, y, z)\) where we have fixed three orthogonal directions and we move \(x\) units in the first direction, \(y\) units in the second direction, and \(z\) units in the third direction.

The \(x\)-axis consists of points of the form \((x, 0, 0)\), the \(y\)-axis consists of points of the form \((0, y, 0)\) and the \(z\)-axis consists of points of the form \((0, 0, z)\). The \(xy\)-plane consists of points of the form \((x, y, 0)\), the \(xz\)-plane consists of points of the form \((x, 0, z)\) and the \(yz\)-plane consists of points of the form \((0, y, z)\). You should be able to sketch a picture of three dimensional space and mark each one of these axes and planes.

Once we can describe position the next step is to measure distance. In two dimensions we can use the Pythagorean Theorem (twice) to get the distance between points \((x_1, y_1)\) and \((x_2, y_2)\) to be

\[ D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \]

On the other hand for three dimensions we will use the Pythagorean Theorem (twice) to get the distance between points \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) to be

\[ D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \]

A sphere is a set of points which are a fixed distance, \(r\), away from a central point, \((h, k, \ell)\). Using the distance formula (where we conveniently square to get rid of the inconvenient square roots) we have that a sphere is the set of points satisfying

\((x - h)^2 + (y - k)^2 + (z - \ell)^2 = r^2.\)

Sometimes we will not have a sphere given to us in this form in which case we should rewrite it (i.e., using complete the square). If we want to have all the points in a solid sphere then we have

\((x - h)^2 + (y - k)^2 + (z - \ell)^2 \leq r^2,\)

the volume of this sphere is \(\frac{4}{3}\pi r^3\).

The midpoint between \((x_1, y_1, z_1)\) and \((x_2, y_2, z_2)\) is the point 

\[ \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right). \]

We can describe motion of a particle by describing position at each time \(t\). This gives us parametric equations \((x(t), y(t), z(t))\). We can take a parametric equation and ask how long is the curve. This can be found by splitting the curve into tiny little pieces, using the distance formula on each piece, and adding them back up (i.e., doing integral calculus). The limit of this process gives us

\[ \text{length} = \int^b_a \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} \, dt. \]

Vectors

Sometimes we are interested in quantities that have both a length and a direction (i.e., velocity or force). We will refer to these as vectors. Most of our discussion will be about three dimensional vectors but most ideas generalize to all dimensions. We note that vectors are not tied to specific points, i.e., they can be translated to any other location and still be the same vector.

Geometrically a vector is a directed line segment and we can add vectors by chaining them one after another, and we can scale vectors by changing the length (note that scaling by a negative number reverses the direction of the vector).

For our purposes it will be convenient to work with a vector in terms of quantities, i.e., algebraically. This is done by writing the vector in component form.

\[ \vec{u} = \langle a, b, c \rangle \]

where \(a\) is the amount of change in the \(x\) direction, \(b\) is the amount of change in the \(y\) direction and \(c\) is the amount of change in the \(z\) direction. As an example, the vector going from the point \((x_1, y_1, z_1)\) to the point \((x_2, y_2, z_2)\) is \((x_2 - x_1, y_2 - y_1, z_2 - z_1)\). In component form we can easily add and scale vectors, working component by component, i.e.,

\[ \langle a, b, c \rangle + \langle d, e, f \rangle = \langle a + d, b + e, c + f \rangle \]

\[ k\langle a, b, c \rangle = \langle ka, kb, kc \rangle. \]

The magnitude of a vector (i.e., the length) can be found in component form by translating the vector so that the tail is at the origin and looking at the distance between the tip of the vector and the origin. In particular we have

\[ ||\langle a, b, c \rangle|| = \sqrt{a^2 + b^2 + c^2}. \]

A vector is a unit vector if it has length 1, any vector that is not the zero-vector \((\vec{0} = \langle 0, 0, 0 \rangle)\) can be scaled to a unit vector by dividing its magnitude, i.e., \(\vec{u}/||\vec{u}||\). This will be used whenever we want to talk about something happening in a particular direction.

Three important unit vectors are \(\vec{i} = \langle 1, 0, 0 \rangle\), \(\vec{j} = \langle 0, 1, 0 \rangle\) and \(\vec{k} = \langle 0, 0, 1 \rangle\). These are known as the standard unit vectors and we can rewrite our vectors as combinations of these three vectors, i.e.,

\[ \langle a, b, c \rangle = \langle a, b, c \rangle + \langle 0, 0, 0 \rangle = a\langle 1, 0, 0 \rangle + b\langle 0, 1, 0 \rangle + c\langle 0, 0, 1 \rangle = a\vec{i} + b\vec{j} + c\vec{k}. \]
**Dot product**

While we can conveniently add and scale vectors there is no convenient way to multiply vectors together. There are two approaches that act like multiplication, we start with the first called the dot product. We have “(vector) · (vector) = number”. In other words the dot product takes two vectors and produces a number. In two and three dimensions this behaves as follows (this works similarly in other dimensions):

\[
(\mathbf{a}, \mathbf{b}) \cdot (\mathbf{a}, \mathbf{b}) = a_1a_2 + b_1b_2
\]

\[
(\mathbf{a}, \mathbf{b}) \cdot (\mathbf{a}, \mathbf{b}) = a_1a_2 + b_1b_2 + c_1c_2
\]

With this definition it is easy to establish some basic facts that make it look like multiplication, i.e.,

\[
\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}, \quad \mathbf{u} \cdot \mathbf{0} = 0, \quad \mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}.
\]

What makes the dot product useful is the geometric interpretation. Let \( \mathbf{u} = (a, b, c) \) then we have

\[
\mathbf{u} \cdot \mathbf{u} = (a, b, c) \cdot (a, b, c) = a^2 + b^2 + c^2 = ||\mathbf{u}||^2
\]

(the same result holds in other dimensions as well). Combining this with the law of cosines we get the following:

\[
\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}|| ||\mathbf{v}|| \cos \theta \quad \text{or} \quad \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{u}|| ||\mathbf{v}||}
\]

This allows us to find angles between vectors. One special angle between vectors is a right angle (90° or \( \frac{\pi}{2} \)). For such an angle we have that \( \cos \theta = 0 \) and this leads us to an easy test of whether two vectors are at right angles to each other. Namely, two vectors are orthogonal or perpendicular if \( \mathbf{u} \cdot \mathbf{v} = 0 \); this follows the convention that the \( 0 \) vector is perpendicular to every other vector. On a side note, we say that two vectors are parallel if they are scalar multiples of one another (this includes the possibility of reversing direction).

This can be used to find the projection of one vector onto another, i.e., “proj \( \mathbf{u} \) \( \mathbf{v} \)” which is the vector \( \mathbf{u} \) projected down onto the vector \( \mathbf{v} \). We have

\[
\text{proj}_\mathbf{v} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{||\mathbf{v}||^2} \right) \mathbf{v}.
\]

Similarly given a force vector \( \mathbf{F} \) and a vector on which the force moves an object along \( \mathbf{D} \) we have that the work is \( w = \mathbf{F} \cdot \mathbf{D} \).

Planes will play an important role in our class. We think of planes as our generalization of lines and when we found lines we needed a point and a slope which we can think of as a point and a direction. When we find a plane we will similarly need a point and a direction (which we can use a vector for). But for direction we don’t want to use a vector in the plane because there are many possible directions. Instead we will use a vector that is perpendicular to the plane, what we call the normal vector. In particular the normal vector \( \mathbf{n} \) is perpendicular to every vector in the plane.

Given a point \((x_0, y_0, z_0)\) and \( \mathbf{n} = (a, b, c) \) then a point \((x, y, z)\) is in the plane if and only if the vector \((x - x_0, y - y_0, z - z_0)\) (which is a vector in the plane) is orthogonal to \( \mathbf{n} \), i.e.,

\[
0 = \mathbf{n} \cdot (x - x_0, y - y_0, z - z_0)
\]

\[
= a(x - x_0) + b(y - y_0) + c(z - z_0)
\]

or rearranging

\[
ax + by + cz = ax_0 + by_0 + cz_0 = d.
\]

In this last form it is easy to read off the normal vector to the plane. In general when we are dealing with planes we will be working with normal vectors. So two planes are parallel when the normal vectors are parallel, the angle between planes is the angle between normal vectors, and so on.

**Review problems**

1. Find the volume for the set of points \((x, y, z)\) satisfying \(2y + 2z - 1 \leq x^2 + y^2 + z^2 \leq 2y + 3\)

2. Find the distance a particle travels along the curve \((e^t, e^{-t}, \sqrt{2}t)\) from \(t = 0\) to \(t = 1\).

3. Find the equation of the sphere which has the line segment joining \((-2, 3, 7)\) and \((4, -1, 5)\) as a diameter.

4. For \(k\) a scalar and \(\mathbf{u}\) a three dimensional vector, prove \(||k\mathbf{u}|| = |k||\mathbf{u}||\).

5. A 40 mph wind is blowing due south (270°), you are on a plane that has bearing of N 60° E (30°), but the plane is traveling due east (0°). What is the airspeed of the plane? How fast is the plane moving relative to the ground?

6. Find the projection of \(\mathbf{u} = -i + 5j + 3k\) onto the vector \(\mathbf{v} = -i + j - 3k\).

7. Find the equation of a plane passing through the point \((2, -3, 1)\) parallel to \(x + 2y - 4z = 2\).

8. Find the equation of the plane corresponding to the set of points \((x, y, z)\) equidistant from the points \((-2, 1, 4)\) and \((6, 3, -2)\).

9. Find \(\cos \theta\) where \(\theta\) is the angle between the planes \(x + 2y - 2z = 17\) and \(4x + 3z = 73\).

10. Find all times \(t\) that the parametric curve given by \((t^3 + e^t + 2, -t + \cos t + 1, 2t + e^t + 2\cos t)\) intersects the plane \(x + 2y - z = 4\).