Applications of Green’s Theorem

Let us suppose that we are starting with a path $C$ and a vector valued function $F$ in the plane. Then as we traverse along $C$ there are two important (unit) vectors, namely $T$, the unit tangent vector $(\frac{dx}{ds}, \frac{dy}{ds})$, and $n$, the unit normal vector $(\frac{du}{ds}, -\frac{dv}{ds})$. So we can consider the following integrals.

\[ \int_C F \cdot T \, ds \quad \text{and} \quad \int_C F \cdot n \, ds. \]

Note that these integrals exist for any $C$, however once we add on the condition that $C$ is a closed curve then we can use Green’s Theorem to simplify the integrals and in particular turn these into double integrals over the region $S$ enclosed by $C$. We have the following:

\[ \oint_C F \cdot n \, ds = \iint_S \left( \frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} \right) \, dA \]
\[ \oint_C F \cdot T \, ds = \iint_S \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \]

The first integral is measuring flow through the curve and is also known as flux.

What makes this interesting is that we can relate the integrals on the right hand side in terms of various operands we can do to $F$ using the $\nabla$ operator. For example the first integral is simply $\nabla \cdot F$ and so is naturally associated with divergence. The second integral is a part of $\nabla \times F$ (where we add a zero component to make $F$ three dimensional), namely it is the part corresponding to the entry in the $z$-direction (i.e., $k$) and we need to pull it off which can be done with a dot product. Updating we now have

\[ \oint_C F \cdot n \, ds = \iint_S (\nabla \cdot F) \, dA \]
\[ \oint_C F \cdot T \, ds = \iint_S (\nabla \times F) \cdot k \, dA \]

In particular, we have the following rule of thumb which will repeat again later.

"$T$" $\leftrightarrow$ curl and "$n$" $\leftrightarrow$ div.

Much of the rest of this chapter is built upon expanding these two basic ideas; in order to get to that point we will first need to work on doing surface integration.

Surface integrals

We have already seen line integrals, where we have a function that we can restrict to points on the line and then integrate along that line. Surface integrals work in the same way. Namely, we have a surface sitting in three-dimensional space and we have a function $f(x, y, z)$ that assigns a value to each point in space, and in particular each point on the surface. We can then integrate this function on the surface by breaking the surface into little tiny pieces "d(SA)", finding the value of the function on each piece, and hence the total contribution of that piece $f(x, y, z) \, d(SA)$, and finally adding all of the little pieces up. In particular we represent our surface integral for the surface $G$ in the form

\[ \text{Surface integral} = \iint_G f(x, y, z) \, d(SA). \]

This is great theory, and intuitive, but we need to see how to accomplish this in practice. We will start with a special case, namely suppose that our surface is formed by a function $z = g(x, y)$ over some region $R$ in the $xy$-plane. Then we can simply express everything in terms of $x$ and $y$ (including our region we integrate over); the hardest part is the "d(SA)" term but we developed that from the last chapter. Therefore we have the following.

\[ \iint_G f(x, y, z) \, d(SA) = \iint_R f(x, y, g(x, y)) \sqrt{(g_x)^2 + (g_y)^2 + 1} \, dA \]

Not surprisingly these types of integrals are generally unpleasant and best avoided. The problem comes from what is happening under the square root.

Amazingly though there is a special case when this simplifies tremendously. Namely if we go back and look at what the flux through the surface is. This works similarly as before and we have

\[ \text{flux} = \iint_G (F \cdot n) \, d(SA). \]

At first glance it would appear that things have gotten worse for us in that the "$n$" term will also involve a square root. But the amazing thing is that these two square roots exactly cancel out. Woohoo! In particular, if we let $F = (M, N, P)$ and have a surface over a region in the plane as before then we have

\[ \iint_G (F \cdot n) \, d(SA) = \iint_R (-Mg_x - Ng_y + P) \, dA. \]

(This assumes that we are dealing with “upward pointing normals”.)

More generally surface integrals can be thought of as being parameterizations of two dimensional regions (similar to the philosophy that a tangent line
is a parameterization of an interval. From this perspective we have

\[ \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)). \]

Using the same approach as used for the Jacobian and surface integrals from the last chapter we have that 

\[ "d(SA) = ||\mathbf{r}_u \times \mathbf{r}_v|| \ dA". \]

This gives us the following more general result where we will let \( S \) denote the region in the \( uv \)-plane

\[ \iint_G f(x, y, z) \ d(SA) = \iint_S f(\mathbf{r}) ||\mathbf{r}_u \times \mathbf{r}_v|| \ dA. \]

(In practice we will not need this more general form for our purposes.)

**Divergence Theorem**

We can compute the flux of a function \( \mathbf{F} \) through any surface. Earlier in the plane we noted that if our curve was closed then we could relate the integral of the flux to the integral of the region it encloses. Amazingly a similar result holds, so similar in that except for some minor notation it is the same. Before we dive into the specifics of what we need and the conclusions of the result, the take home message is that the flux can be computed by either an integral on the boundary dotted with the normal or an integral on the interior of the divergence in both two and three dimensions.

Also before we get too far we should talk about notation. In an effort to be green before it was cool, mathematicians started to reuse symbols. We will be looking at solid shapes and talking about the bounding surfaces, in this case if \( S \) is our solid then we will denote the boundary of \( S \) (i.e., the surface on the exterior of \( S \)) as \( \partial S \). This uses the same "\( \partial \)" that we saw in partial derivatives. So to be clear, if the "\( \partial \)" is in front of a shape we read this as the boundary of that shape and if the "\( \partial \)" is in front of a function we read this as the partial derivative of that function.

The Divergence Theorem then states that given a solid shape \( S \) with a boundary \( \partial S \) that is "nice" (i.e., composed of pieces that are smooth, a fancy way of saying we can do calculus with those pieces) and we have a function \( \mathbf{F} = \langle M, N, P \rangle \) with nice partial derivatives then

\[ \text{Flux} = \iint_{\partial S} (\mathbf{F} \cdot \mathbf{n}) \ d(SA) = \iiint_S (\nabla \cdot \mathbf{F}) \ dV. \]

The normal vector \( \mathbf{n} \) will always point out, i.e., away from the interior. Note that the conclusion to this theorem still holds for cases when the boundary surface is composed of several pieces.

In practice this is useful because we can take a hard surface integral and reduce it to a less-hard integral over a shape. In the latter case we can often use some basic tools to compute this integral (i.e., volumes, symmetry and so on).

**Stokes’s Theorem**

The generalization of the other result is known as Stokes’s Theorem. Before diving into it let us first note that while we can talk about normals to both curves and surfaces in a meaningful way, there is no way to talk about tangent to a surface in a meaningful way. So when we generalize the result relating to the tangent vector we will essentially stay in the same setting where we have a single region (now a surface) with a boundary and we are computing an integral over both the surface and the boundary of the surface. Viewed in this context the result is nearly identical to what we had before.

Given a function \( \mathbf{F} = \langle M, N, P \rangle \) with nice partial derivatives and a surface \( S \) which is smooth with smooth boundary \( \partial S \) then

\[ \oint_{\partial S} \mathbf{F} \cdot \mathbf{T} \ ds = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \ d(SA). \]

The hardest part about applying this is getting the orientation on the curve correct. Given \( \mathbf{n} \) imagine that we slice off a thin strip next to \( \partial S \) and that we will walk along \( \partial S \) on this thin strip standing with our head in the direction of \( \mathbf{n} \). Then the direction that we should travel along the curve is the one which will keep the surface to our left hand side. “To the left, to the left, always keep the surface on the side to the left.” Conversely if we have an orientation on our curve, this gives an orientation to the surface, i.e., \( \mathbf{n} \), by a similar argument.

This is generally a hard theorem to use for a few reasons. First off we have a hard time motivating what is going on except to vaguely way that this is somehow measuring rotation. Second off both sides of the above equation can be very difficult to compute so we do not seem to gain much. But there is a secret that can make many of these problems easier. And once you see the secret then the theorem becomes many times more amazing. Here it is:

**Different surfaces \( S \) and \( T \) can have \( \partial S = \partial T \).**

Why is this useful? Because the integral on the left only looks at \( \partial S \), so this says that we can reduce it to a surface which better fits our mood. Generally that means we can reduce it to a surface that is flat. For such a surface it is easy to then find \( \mathbf{n} \) and the problem becomes many factors of time easier! The majority of Stokes’s Theorem problems can be done quickly and easily by using this observation.