Enumeration of Words with Forbidden Patterns

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Abstract

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For any $j \in \mathbb{N}$, define $[j]$ to be a totally ordered set $\{1, 2, \ldots, j\}$. A word $\sigma \in [k]^n$ (i.e. an n-letter word over a totally ordered k-letter alphabet) *avoids* a word $\tau \in [\ell]^m$ iff there is no $1 \leq i_1 < \ldots < i_m \leq n$ such that $(\sigma(i_1), \ldots, \sigma(i_m))$ has exactly ℓ distinct letters and is order-isomorphic to $\tau = (\tau(1), \ldots, \tau(m))$. In other words, there is no subsequence of σ of length m (which is the length of τ) such that it has the same pairwise comparisons of letters as τ. In this case, we write $\sigma \in [k]^n(\tau)$ and call the word τ a *pattern*. If a word σ avoids all patterns τ in a set T, we say that σ avoids T and write $\sigma \in [k]^n(T)$. We find the cardinality of $[k]^n(\tau)$ for various τ as well as of $[k]^n(T)$ for all subsets T of S_3. We also establish some and conjecture other equivalence relations φ on S_ℓ (so $m = \ell$ in this case) other than reversal, induced by $r : \tau(j) \to \tau(m - j + 1)$, and complementation, induced by $c : \tau(j) \to \ell - \tau(j) + 1$, such that for any $\ell, k, n \in \mathbb{N}$ and any $\tau_1, \tau_2 \in S_\ell$, $\tau_1 \varphi \tau_2$ iff $|[k]^n(\tau_1)| = |[k]^n(\tau_2)|$.

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Introduction

Chapter 1

A survey of related literature

In this dissertation, we will focus on enumeration of words with forbidden patterns, a topic which has been almost unexplored up to now (currently, the sole exception is Regev [17]), yet very closely related to enumeration of permutations with forbidden patterns, a part of the study of occurrences of patterns in permutations, an area in combinatorics studied intensively and extensively in recent years.

In this chapter, we will provide an overview of the related literature on permutations with forbidden patterns, and devote the next chapter to the overview of the main results of this thesis.

1.1 Patterns in words and permutations

We will start with the definition of a pattern.

Definition 1.1 *Let $[k] = \{1, \ldots, k\}$ be a (totally ordered) alphabet on k letters, and let $\sigma \in [k]^n$, $\tau \in [\ell]^m$, $\ell \leq k$, $m \leq n$. Also let τ contain all letters in $[\ell]$. An occurrence of τ in σ is a choice of m positions $1 \leq i_1 < \ldots < i_m \leq n$, such that the set $\{\sigma(i_1), \ldots, \sigma(i_m)\}$ contains exactly ℓ distinct letters and $(\sigma(i_1), \ldots, \sigma(i_m))$ is order-isomorphic to $(\tau(1), \ldots, \tau(m))$. In this situation, the word τ is called a pattern.*

Example 1.1 $\sigma = 5234351 \in [5]^7$ contains occurrences of pattern $\tau = 123 \in [3]^3$ e.g., $5\underline{2}3\underline{4}3\underline{5}1$. Here $(i_1, i_2, i_3) = (2, 4, 6)$, and $(\sigma(i_1), \sigma(i_2), \sigma(i_3)) = (2, 4, 5)$ is order isomorphic to $(\tau(1), \tau(2), \tau(3)) = (1, 2, 3)$. Similarly, σ also contains an occurrence of the pattern $\tau' = 212$ (e.g., $\underline{5}2\underline{3}43\underline{5}1$).

Remark 1.1 When σ and τ are permutations (i.e. $n = k$, $\ell = m$, $\sigma \in S_n$, $\tau \in S_m$), our definition of occurrence is equivalent to requiring that

$$(\sigma_{\tau^{-1}(1)}, \ldots, \sigma_{\tau^{-1}(m)})$$

be increasing.

DEFINITION 1.2 *If σ does not contain any occurrences of τ, we say that σ avoids τ. If σ avoids all patterns in a set T we say that σ avoids T.*

NOTATION 1.1

$$[k]^n(\tau) = \{\sigma \in [k]^n : \sigma \text{ avoids } \tau\}$$

$$[k]^n(T) = \{\sigma \in [k]^n : \sigma \text{ avoids all } \tau \in T\} = \bigcap_{\tau \in T} [k]^n(\tau)$$

$$S_n(\tau) = \{\sigma \in S_n : \sigma \text{ avoids } \tau\}$$

$$S_n(T) = \{\sigma \in S_n : \sigma \text{ avoids all } \tau \in T\} = \bigcap_{\tau \in T} S_n(\tau)$$

REMARK 1.2 Suppose that the shortest pattern in T has length m, and each pattern has at least ℓ distinct letters. Then, if either $k < \ell$ or $n < m$, we have $[k]^n(T) = [k]^n$, $S_n(T) = S_n$.

1.2 Sorting through queues and stacks

Donald Knuth ([11], vol. 1, section 2.2.1) originally stimulated interest in the study of occurrences of patterns in permutations by demonstrating its connection to sorting sequences through a given arrangement of queues and stacks (say, sorting railroad cars through a given switchyard). In some instances, the sortability of a sequence depends on whether or not it avoids a certain pattern.

The trivial example is when a switchyard contains only 1 queue, i.e. when there is a single track with no switchyard. In this case, a sortable sequence is one which is already sorted, i.e. avoids the pattern 21.

The simplest nontrivial example is a switchyard with only 1 stack (together with the entrance queue (with the original sequence) and the exit queue (with the resulting sequence)).

We remind the reader that a stack functions in the FILO mode (first in, last out), while a queue operates in the FIFO mode (first in, first out). In other words, at any step, the only element which may be moved out of a stack is the one added last, while the only element which may be moved out of a queue is the one added earliest.

THEOREM 1.1 (KNUTH) *A permutation σ is 1-stack sortable iff σ avoids 231.*

The ($\Longrightarrow$) part is easy to prove. One sees immediately that the sequence 231 cannot be sorted through one stack, and hence, neither can any occurrence of the pattern 231 in σ. To prove the ($\Longleftarrow$) part, we show that 1 in σ can be moved through a stack while preserving the sequential order of the rest of the elements in σ, and then proceed by induction. The crucial observation here is that if σ avoids the pattern 231, then the substring of σ preceding 1 must be non-increasing.

REMARK 1.3 Note that the same result and proof apply in the case of 1-stack sortability of a given word.

EXAMPLE 1.2 The permutation 53241 cannot be sorted through a stack since it contains an occurrence of 231, say, at positions $(2, 4, 5)$: $5\underline{3}2\underline{41}$. On the other hand, the permutation 14235 can indeed be sorted through a stack since it avoids the pattern 231.

Robert Tarjan continued the study of railroad switchyards in [26], finding further connections between sortability of a sequence and the occurrence of a certain pattern in the corresponding permutation.

THEOREM 1.2 (TARJAN) *A permutation σ is sortable through ℓ parallel stacks iff σ avoids the pattern $2\,3\ldots(\ell+2)\,1$.*

THEOREM 1.3 (TARJAN) *A permutation σ is sortable through ℓ parallel queues iff σ avoids the pattern $(\ell+1)\,\ell\ldots1$.*

THEOREM 1.4 (TARJAN) *A permutation σ is sortable through ℓ parallel stacks subject to the condition that all cars are moved into the stacks before any of the cars are moved from the stacks iff σ avoids the pattern $1\,2\ldots(\ell{+}1)$.*

1.3 Symmetries

Here we will make some symmetry arguments, which we will often use later in this thesis. There are two symmetries on words: *reversal* and *complementation.* Permutations also have an additional symmetry induced by the *inverse* operation.

DEFINITION 1.3 *We define the reversal $r : [\ell]^m \to [\ell]^m$ and complementation $c : [\ell]^m \to [\ell]^m$ as follows:*

$$\tau' = r(\tau) \iff (\forall j \in [m])(\tau'(j) = \tau(m - j + 1)),$$

$$\tau' = c(\tau) \iff (\forall j \in [m])(\tau'(j) = \ell - \tau(j) + 1).$$

We can similarly define reversal and complementation on S_ℓ, noting that $m = \ell$ in this case.

EXAMPLE 1.3

(i) For $\sigma = 5234351 \in [5]^7$ of Example 1.1, we have $r(\sigma) = 1534325$ and $c(\sigma) = 1432315$.

(ii) For $\sigma = 1342 \in S_4$, we have $r(\sigma) = 2431$, $c(\sigma) = 4213$, $\sigma^{-1} = 1423$.

We can obtain the following directly from the definitions of reversal, complementation and inverse.

FACT 1.1 *The group G_w (resp. G_p) of transformations of $[k]^n$ (resp. S_n) generated by r and c (resp. r, c and the inverse $i = \langle \cdot \rangle^{-1}$) is isomorphic to the dihedral group D_4 (resp. D_8).*

Proof. Notice that $r^2 = c^2 = (r \cdot c)^2 = 1$ (hence $r \cdot c = c \cdot r$), which ends the proof for G_w. Also, $i^2 = (r \cdot i)^4 = (c \cdot i)^4 = 1$ and $i \cdot r \cdot i = c$, so $H = \langle r \cdot i, i \rangle$ and the rest is easy to check. □

FACT 1.2 *Given a pattern $\tau \in [\ell]^m$, a word $\sigma \in [k^n]$ and a choice of positions $1 \leq i_1 < \ldots < i_m \leq n$, the following are equivalent:*

(i) τ occurs in σ at positions $1 \leq i_1 < \ldots < i_m \leq n$.

(ii) $c(\tau)$ occurs in $c(\sigma)$ at positions $1 \leq i_1 < \ldots < i_m \leq n$.

(iii) $r(\tau)$ occurs in $r(\sigma)$ at positions $1 \leq n - i_m + 1 < \ldots < n - i_1 + 1 \leq n$.

(iv) if both σ and τ are permutations, τ^{-1} occurs in σ^{-1} at positions $1 \leq \sigma(i_{\tau^{-1}(1)}) < \ldots < \sigma(i_{\tau^{-1}(m)}) \leq n$ (see Remark 1.1).

The next statement immediately follows from the preceding one.

FACT 1.3 *The following are equivalent:*

(i) σ avoids τ.

(ii) $c(\sigma)$ avoids $c(\tau)$.

(iii) $r(\sigma)$ avoids $r(\tau)$.

(iv) if both σ and τ are permutations, σ^{-1} avoids τ^{-1}.

The Facts 1.2 and 1.3 invite the following definition.

DEFINITION 1.4 *Given a pattern τ, let $\bar{\tau}^w$ (resp. $\bar{\tau}^p$), the* symmetry class *of τ on words (resp. on permutations) be the set of all patterns which can be obtained from τ by an action of G_w (resp. G_p).*

We will usually drop the superscript when it is clear which one is meant or that both are meant.

The symmetry classes of S_3 and S_4 on words are listed in Tables 1.1 and 1.2. On permutations, the symmetry classes of S_3 are the same as on words,

Symmetry class	Patterns
$\overline{123}$	123, 321
$\overline{132}$	132, 213, 231, 312

Table 1.1: Symmetry classes for S_3 on words

while on S_4 the only exception is $\overline{1342}^p = \overline{1342}^w \cup \overline{1423}^w$ since 1423 is the inverse of 1342. We can see that S_3 has 2 symmetry class both on words and on permutations, and S_4 has 8 symmetry classes on words and 7 symmetry classes on permutations.

1.4 Cardinalities and Wilf classes

Notation 1.2 We denote the cardinality a set S by $|S|$.

As an immediate corollary of Fact 1.3, we have

Fact 1.4 *If $\tau_1, \tau_2 \in [\ell]^m$ are in the same symmetry class on words, then $|[k]^n(\tau_1)| = |[k]^n(\tau_2)|$ for any $k, n \in \mathbb{N}$. If $\tau_1, \tau_2 \in S_m$ are in the same symmetry class on permutations, then $|S_n(\tau_1)| = |S_n(\tau_2)|$ for any $n \in \mathbb{N}$.*

Remark 1.4 Notice that we can apply reversal, complementation or inverse to a set of patterns by applying it to each pattern in the set, and obtain the

Symmetry class	Patterns
$\overline{1234}$	1234, 4321
$\overline{1243}$	1243, 2134, 3421, 4312
$\overline{1324}$	1324, 4231
$\overline{1342}$	1342, 2431, 3124, 4213
$\overline{1423}$	1423, 2314, 3241, 4132
$\overline{1432}$	1432, 2341, 3214, 4123
$\overline{2143}$	2143, 3412
$\overline{2413}$	2413, 3142

Table 1.2: Symmetry classes for S_4 on words

analogous fact for sets of patterns.

The earliest to be examined was the case of permutations avoiding patterns of length 3. Knuth ([11], vol. 1, sec. 2.2.1) found the exact formula for $\tau \in \overline{132}$ and showed ([11], vol. 3, sec. 5.1.4) that the same formula holds for $\tau \in \overline{123}$ by using Schensted's [19] correspondence between permutations and Young tableaux and MacMahon's ([12], vol1, Chapter III.V) earlier result on Young tableaux.

THEOREM 1.5 (KNUTH) *For any $\tau \in S_3$, $|S_n(\tau)| = c_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number.*

In particular, railroad switchyard Theorems 1.1, 1.2 and 1.3 (for $\ell = 2$), together with the facts that $|S_n(321)| = c_n$, $|S_n(231)| = c_n$ and $|S_n(123)| = c_n$, respectively, give us the following corollaries.

COROLLARY 1.1 *There are c_n permutations which can be sorted through 2 parallel queues.*

COROLLARY 1.2 *There are c_n permutations which can be sorted through 1 stack.*

COROLLARY 1.3 *There are c_n permutations which can be sorted through 2 parallel stacks subject to the restriction that all the numbers must be moved into the stacks (from the entrance queue) before any can be moved out of the stacks (into the exit queue).*

In Knuth's book [11], the two results which together make up Theorem 1.5 were obtained in different ways without a direct bijection between, say, $S_n(132)$ and $S_n(123)$. The first such bijection was given by Rodica Simion and Frank Schmidt [21]. Later, other bijective proofs were given by Dana Richards [18] and Julian West [27].

Definition 1.5 *Two sets of patterns, T_1 and T_2, are said to be Wilf-equivalent on words and to belong to the same Wilf*[1] *class* (or *cardinality class*) *on words iff*

$$(\forall k, n \in \mathbb{N})\,(|[k]^n(T_1)| = |[k]^n(T_2)|).$$

Definition 1.6 *Two sets of patterns, T_1 and T_2, are said to be Wilf-equivalent on permutations and to belong to the same Wilf class on permutations iff*

$$(\forall n \in \mathbb{N})\,(|S_n(T_1)| = |S_n(T_2)|).$$

Notation 1.3 If T_1 and T_2 are Wilf-equivalent on words, we write $T_1 \sim_w T_2$. If T_1 and T_2 are Wilf-equivalent on permutations, we write $T_1 \sim_p T_2$. We also denote the Wilf class on words of T by $\boxed{T}_w$, and on permutations, by $\boxed{T}_p$. Again, we will drop the subscript if we know which one is meant or that both are meant.

Most research on counting permutations avoiding patterns concentrated on single patterns, i.e. on sets of one pattern, of the same length. However, there are some exceptions. Simion and Schmidt [21] found the cardinali-

[1]Herb Wilf raised the question of when it is true that two sets of forbidden patterns are equally restrictive in the early 1980s.

ties of $|S_n(T)|$ for any $T \subseteq S_3$, and West [29], [28] proved bijectively that $|S_n(3142, 2413)| = |S_n(1423, 1324)| = s_n$, the nth Schröder number, given by $s_n = \sum_{i=0}^{n} \binom{n+i}{n-i} c_i$, by making use of generating trees introduced in [7]. Also, there are some results on the number of 3,4-restricted permutations (i.e. avoiding a set of patterns $T = \{\tau_1, \tau_2\}$ with $\tau_1 \in S_3$ and $\tau_2 \in S_4$). Those results are listed in West [29], p. 372.

We list the results of Simion and Schmidt in Table 1.3. F_{n+1} there is the $(n+1)$-st Fibonacci number.

For patterns of greater length, the cardinalities $|S_n(T)|$ become much more difficult to compute, even for $T \subset S_4$. This is especially true when $|T| \ll |S_m| = m!$, in particular, for $T = \{\tau\}$. In this case, a better use of our effort may be to determine the Wilf classes (i.e. the cardinality classes) of patterns, as well as to find bounds and asymptotic results for cardinalities of sets avoiding a given set of patterns. To some degree, this may be due to the fact that the problem of deciding whether a given pattern τ occurs in a given permutation or word σ is NP-complete.

For reasons mentioned above, it is easier to look only at some specific patterns of general length, mainly the identity pattern. Two asymptotic results in this direction belong to Amitai Regev [16], [17].

Theorem 1.6 (Regev) *Let $id_{\ell+1} \in S_{\ell+1}$ be the identity permutation of length $\ell+1$. Then*

$$|S_n(id_{\ell+1})| \simeq c(\ell)\frac{\ell^{2n}}{n^{(\ell^2-1)/2}}\,,\ \text{where } c(\ell) = \frac{\ell^{\ell^2/2}\cdot\prod_{j=1}^{\ell-1} j!}{(2\pi)^{(\ell-1)/2}\cdot 2^{(\ell^2-1)/2}} \tag{1.1}$$

as $n \to \infty$.

Theorem 1.7 (Regev) *For any $k, \ell \in \mathbb{N}$,*

$$|[k]^n(id_{\ell+1})| \simeq \frac{0!1!2!\dots(\ell-1)!}{(k-\ell)!\dots(k-1)!}\cdot\left(\frac{1}{\ell}\right)^{\ell(k-\ell)}\cdot n^{\ell(k-\ell)}\cdot \ell^n \tag{1.2}$$

as $n \to \infty$.

Also, note, after Regev, that

$$\frac{0!1!2!\dots(\ell-1)!}{(k-\ell)!\dots(k-1)!} = \left(\frac{k!!}{\ell!!(k-\ell)!!}\right)^{-1}, \tag{1.3}$$

where we denote $m!! = 0!1!\dots(m-1)!$.

Note that, as a consequence of Stirling's formula, we have for $\ell = 3$ that

$$|S_n(123)| = \frac{1}{n+1}\binom{2n}{n} \simeq \frac{1}{\sqrt{\pi}}\frac{4^n}{n^{3/2}}$$

which agrees with the Theorem 1.6. We will see later that our exact result for $|[k]^n(123)|$ also agrees with Theorem 1.7.

Another general result, involving generating functions, belongs to Ira Gessel [8].

THEOREM 1.8 (GESSEL) *Let $id_\ell \in S_\ell$ be the identity permutation of length ℓ, $u_\ell(n) = |S_n(id_\ell)|$. Then*

$$U_\ell(x) = \sum_{n=0}^{\infty} u_\ell(n) \frac{x^{2n}}{n!} = \det(I_{|i-j|}(x))_{1 \le i,j \le \ell}, \tag{1.4}$$

where $I_i(x)$ is a Bessel function

$$I_i(X) = \sum_{n=0}^{\infty} \frac{x^{2n+i}}{n!(n+i)!} = \sum_{n=0}^{\infty} \binom{2n+i}{n} \frac{x^{2n+i}}{(2n+i)!}.$$

This result was later explained in terms of lattice walks by Ira Gessel, Jonathan Weinstein and Herbert Wilf in [9].

We will now look at permutation patterns of length 4 which were studied in more detail. Their study was begun by Julian West [27], who proved the following theorem.

THEOREM 1.9 (WEST) *$S_n(1234)$ is in bijection with $S_n(1243)$, which is in bijection with $S_n(2143)$, so $|S_n(1234)| = |S_n(1243)| = |S_n(2143)|$, and hence, $\overline{1234}$, $\overline{1243}$ and $\overline{2143}$ belong to the same Wilf class.*

The determination of Wilf classes of S_4 on permutations was completed by Zvezdelina Stankova [22], [23].

THEOREM 1.10 (STANKOVA, [22]) *$S_n(4132)$ is in bijection with $S_n(3142)$, so $|S_n(4132)| = |S_n(3142)|$, and hence, $\overline{1342}$ and $\overline{2413}$ belong to the same*

Wilf class.

THEOREM 1.11 (STANKOVA, [23]) *$S_n(1234)$ is in bijection with $S_n(4123)$, so $|S_n(1234)| = |S_n(4123)|$, and hence, $\overline{1234}$ and $\overline{1432}$ belong to the same Wilf class.*

We remind the reader that 1423 is an inverse of 1342, and hence, $\overline{1342} = \overline{1423}$ on permutations.

The numerical results in [27] (also cited in [23]), namely,

$$S_7(1342) = 2740 < S_7(1234) = 2761 < S_7(1324) = 2762, \qquad (1.5)$$

complete the classification of Wilf classes of S_4. However, the results of Miklós Bóna in [1] and [5] (smaller upper bound for $|S_n(1324)|$ in the latter article), show that more is true.

THEOREM 1.12 (BÓNA) *$|S_n(1423)| \leq |S_n(1234)| \leq |S_n(1324)| < 25^n$ for all $n \in \mathbb{N}$, and, in fact, $|S_n(1423)| < |S_n(1234)| < |S_n(1324)| < 25^n$ for all $n \geq 7$.*

In other words, on permutations, $\boxed{1423}$ is *asymptotically smaller* than $\boxed{1234}$, which, in turn, is asymptotically smaller than $\boxed{1324}$.

Of the exact cardinalities of the 3 Wilf classes of S_4 on permutations, only $|S_n(1324)|$ is not known. As a corollary of his general theorem (Theorem 1.8), Gessel [8] found the exact value of $|S_(1234)|$.

Corollary 1.4 (Gessel)

$$|S_n(1234)| = u_4(n) = 2\sum_{i=0}^{n} \binom{2i}{i}\binom{n}{i}^2 \frac{3i^2+2i+1-n-2in}{(i+1)^2(i+2)(n-i+1)} \tag{1.6}$$

The only exact result for a pattern not in the Wilf class of the identity permutation belongs to Miklós Bóna [4].

Theorem 1.13 (Bóna)

$$\begin{aligned}|S_n(1342)| = (-1)^{n-1}\,\frac{7n^2-3n-2}{2}&+\\ &+3\sum_{i=2}^{n} 2^{i+1}\frac{(2i+4)!}{i!(i-2)!}\binom{n-i+2}{2}(-1)^{n-i}.\end{aligned} \tag{1.7}$$

Corollary 1.5 (Bóna)

$$\lim_{n\to\infty} |S_n(1342)|^{\frac{1}{n}} = 8. \tag{1.8}$$

Note that Theorem 1.6 implies that

$$\lim_{n\to\infty} |S_n(id_{\ell+1})|^{\frac{1}{n}} = \ell^2, \tag{1.9}$$

so

$$\lim_{n\to\infty} |S_n(1234)|^{\frac{1}{n}} = 9. \tag{1.10}$$

Therefore, we see that $\lim_{n\to\infty} |S_n(\tau)|^{\frac{1}{n}}$, if it exists, depends on more than just the length of τ. As noted in [15], it is interesting that this limit is equal to an integer in all cases where it exists and its value is known.

However, even the existence of such a limit is not proven for a general pattern τ. Herbert Wilf [33] and Richard Stanley independently made the following conjecture.

CONJECTURE 1.1 (STANLEY, WILF) *For any pattern* τ, $\lim_{n\to\infty} |S_n(\tau)|^{\frac{1}{n}}$ *exists as a positive finite number.*

In addition, Wilf [33] proved the following theorem:

THEOREM 1.14 (WILF) *For any pattern* τ, $\liminf_{n\to\infty} |S_n(\tau)|^{\frac{1}{n}} > 0$, *thus, if* $\lim_{n\to\infty} |S_n(\tau)|^{\frac{1}{n}}$ *exists, it must be positive.*

Note that all the special cases considered so far support the Stanley-Wilf conjecture. A significant advance toward proving it has recently been made by Bóna [5].

DEFINITION 1.7 *A layered pattern is a pattern which can partitioned into substrings (layers) so that the elements are decreasing within layers and increasing between layers.*

EXAMPLE 1.4 In S_4, the layered patterns are $\hat{1}\hat{2}\hat{3}\hat{4}$ and $\hat{1}\widehat{32}\hat{4}$. Their layers are denoted by hats.

THEOREM 1.15 (BÓNA) *For all $m \in \mathbb{N}$, there exists a constant $c = c(m)$, such that for all layered patterns $\tau \in S_m$, $|S_n(\tau)| < c^n$.*

1.5 A general counting problem; P-recursiveness

We will now look at our counting problem in a slightly more general context. Namely, instead of looking at permutations or words with forbidden patterns, we consider permutations or words with *distinguished* patterns. In other words, we are given a list of patterns $T = (\tau_i)_{i \in I}$ and a vector of occurrences $\mathbf{r} = (r_i)_{i \in I}$, and we want to determine $|S_n(T; \mathbf{r})|$ or $|[k]^n(T; \mathbf{r})|$, i.e. the cardinality of the set of permutations or words which simultaneously contain each $\tau_i \in T$ exactly r_i times.

REMARK 1.5 Note that symmetry classes of distinguished sets of patterns are the same as those of forbidden sets of patterns, i.e. do not depend on $\mathbf{r}$ (which is $\mathbf{0}$ if all patterns in a set are forbidden).

Most of the research in this direction concentrated on single distinguished patterns, that is, on $T = \{\tau\}$ and $\mathbf{r} = \{r\}$.

John Noonan [13] and Miklós Bóna [2] obtained exact formulas for the number of permutations of containing exactly one occurrence of 123 and 132, respectively.

Theorem 1.16 (Noonan) $|S_n(123;1)| = \frac{3}{n}\binom{2n}{n-3}$.

Theorem 1.17 (Bóna) $|S_n(132;1)| = \binom{2n-3}{n-3}$.

As noted in Price [15], although $|S_n(123;0)| = |S_n(132;0)|$, the same is not true for $|S_n(123;1)|$ and $|S_n(132;1)|$. Moreover, even the monotonicity analogous to that in Theorem 1.12 does not hold, since it is not hard to see that $|S_n(123;1)| > |S_n(132;1)|$ for $4 \le n \le 14$ and $|S_n(123;1)| < |S_n(132;1)|$ for $n \ge 15$.

In general, it appears quite hard to even prove that the coefficients in the recurrence relation for $|S_n(\tau;r)|$ are rational functions of n.

Definition 1.8 *We call a sequence (x_n) polynomially recursive, or P-recursive, if there exists $t \in \mathbb{N}$ and polynomials $p_0(n), \ldots, p_t(n)$ such that for all $n \in \mathbb{N}$,*

$$p_0(n)x_n + p_1(n)x_{n+1} + \cdots + p_t(n)x_{n+t} = 0.$$

Again, the identity pattern was first to be studied. The following result was obtained by Gessel [8].

THEOREM 1.18 (GESSEL) *For any given $m \in \mathbb{N}$, the sequence $(|S_n(id_m)|)$ is P-recursive in n.*

The first two (and so far the only) results on non-identity patterns were obtained by Bóna [3], [4].

THEOREM 1.19 (BÓNA) *For any given $r \in \mathbb{N}$, the sequence $(|S_n(132;r)|)_{n=0}^{\infty}$ is P-recursive.*

THEOREM 1.20 (BÓNA) *The sequence $(|S_n(1342)|)_{n=0}^{\infty}$ is P-recursive.*

Noonan and Zeilberger [14] have made the following general conjecture.

CONJECTURE 1.2 (NOONAN, ZEILBERGER) *For any fixed pattern τ and $r \in \mathbb{N}$, the sequence $(|S_n(\tau;r)|)_{n=0}^{\infty}$ is P-recursive.*

This conjecture follows from a general method for finding polynomial recurrences for any τ and r described in the same paper which we also will use extensively in the next part of this thesis. We will describe it in detail in the next section. In fact, we note that the method applies for distinguished sets of patterns as well, and hence, we may generalize the preceding conjecture slightly.

Conjecture 1.3 *For any fixed set of patterns T and vector of occurrences $\mathbf{r}$, the sequence $(|S_n(T;\mathbf{r})|)$ is P-recursive in n.*

1.6 Noonan-Zeilberger method

We will now outline the method described in [14] adding a little more generality by looking at sets of patterns rather than single patterns. We will also add the corrections needed to apply it on words (rather than on permutations) with distinguished set of patterns.

Our goal is to find $|S_n(T;\mathbf{r})|$ or $|[k]^n(T;\mathbf{r})|$ given a list of patterns T, a vector $\mathbf{r}$ of their occurrences and $n \in \mathbb{N}$. The method is as follows.

Method:

1. Determine the best way to obtain a recurrence for T. Usually, remove
   - the first letter or the last letter of the permutation (word); or
   - the least or the greatest letter of the permutation (word) (leftmost or rightmost one, on words).
2. Identify auxiliary parameters of the recurrence. Usually, these arise from the auxiliary subpatterns which arise after Step 1. This is best

clarified by an example which will follow after the description.

3. Assign the a new main variable for each pattern in T, and then a new auxiliary variable for each auxiliary parameter.

4. Define *weight* of permutation (word) σ.

$$wt(\sigma) = \prod q_\tau^{\varphi_\tau(\sigma)},$$

where the product is over all patterns (main and auxiliary) which arise after Step 1, q_τ is the variable assigned to (sub)pattern τ, and $\varphi_\tau(\sigma)$ is the number of occurrences of τ in σ.

5. Let

$$P_n = \sum_{\sigma \in S_n} wt(\sigma).$$

(In our case, i.e. for words avoiding permutations, we let

$$P_{n,s} = \sum_{\{\sigma \in [k]^n \,:\, s(\sigma)=s\}} wt(\sigma),$$

where $s(\sigma) = (s_1(\sigma), \ldots, s_k(\sigma))$ is the *supply vector*, $s_i(\sigma)$ being the supply of letter i, i.e. the number of occurrences of i in σ.)

6. Using the recurrence from Step 1, determine the functional equation.

7. Take (r_τ)-th derivative with respect to q_τ for each main pattern τ, i.e. for each $\tau \in T$.

8. Let $q_\tau = 0$ for all patterns $\tau \in T$ and $q_\tau = 1$ for all auxiliary patterns τ.

9. For words avoiding permutations, sum the resulting equation over all supplies s such that $|s| = s_1 + \ldots + s_k = n$.

Example 1.5 We determined $|[k]^n(123)|$ by applying the Noonan-Zeilberger method (see section 3.1 of this thesis).

Example 1.6 (Noonan, Zeilberger) To find $|S_n(123, 213)|$ by applying the Noonan-Zeilberger method to find its recurrence relation. In this case, $T = (123, 213)$, $\mathbf{r} = (0, 0)$. Given a permutation $\sigma \in S_n$.

1. Remove the last letter $\sigma(n) = i$ of σ.

2. Look for the auxiliary subpatterns which recursively arise after Step 1. Let $\sigma' = \sigma(1) \ldots \sigma(n-1)$. Then it is easy to see that σ avoids 123 iff σ' avoids 123 and occurrences of 12 with "2" $< i$ (i.e. subsequences $i_1 i_2$ with $i_1 < i_2 < i$). Similarly, σ avoids 213 iff σ' avoids 213 and occurrences of 21 with "2" $< i$ (i.e. subsequences $i_1 i_2$ with $i_2 < i_1 < i$). Thus, in both cases, the auxiliary patterns 12 and 21 must specify a value for the "2," so we need to introduce an auxiliary variable for each

subpattern of the type 12, “2” $= j < i$, and for each subpattern of the type 21, “2” $= j < i$.

Now we consider the new auxiliary patterns 12 and 21 similarly. σ contains an occurrence of 12 iff σ' contains either 12 or a letter $j < i$ (i.e. an occurrence of pattern 1 with “1” $< i$). σ contains an occurrence of 21 iff σ' contains either 21 or a letter $j > i$ (i.e. an occurrence of pattern 1 with “1” $> i$). For subpatterns of length 1, we do not assign auxiliary variables.

3. Assign variables. Main variables:

$$123 \mapsto q, \qquad 213 \mapsto \xi,$$

$$12(\text{“2”} = j) \mapsto q_j \quad 21(\text{“2”} = j) \mapsto \xi_j \qquad (j = 2, \ldots, n).$$

Also, let

- $\varphi(\sigma)$ be the number of occurrences of 123 in σ,
- $\psi(\sigma)$, the number of occurrences of 213 in σ,
- $\varphi_j(\sigma)$, the number of occurrences of 12, “2” $= j$, in σ, where $j = 2, \ldots, n$,
- $\psi_j(\sigma)$, the number of occurrences of 21, “2” $= j$, in σ, where $j = 2, \ldots, n$.

4. Define the weight of σ:

$$wt(\sigma) = q^{\varphi(\sigma)}\xi^{\psi(\sigma)}\prod_{j=2}^{n}\left(q_j^{\varphi_j(\sigma)}\xi_j^{\psi_j(\sigma)}\right)$$

5. Define

$$P_n(q;\xi;q_2,\ldots,q_n;\xi_2,\ldots,\xi_n) = \sum_{\sigma\in S_n} wt(\sigma) = \sum_{i=1}^{n}\sum_{\substack{\sigma\in S_n\\ \sigma(n)=i}} wt(\sigma)$$

6. Determine the recursive functional equation for P_n. This follows from the recursive equation for $wt(\sigma)$ summed over all σ. If $\sigma(n) = i$, then

$$wt(\sigma) = wt(\sigma')\prod_{j=2}^{i-1}\left((qq_j)^{\varphi_j(\sigma')}(\xi\xi_j)^{\psi_j(\sigma')}\right) q_i^{i-1}\prod_{j=i+1}^{n}\xi_j$$

Summing over all σ with $\sigma(n) = i$, then over all $i = 1, 2, \ldots, n$, we get

$$\begin{aligned} P_n(q;\xi;q_2,\ldots,q_n;\xi_2,\ldots,\xi_n) &= \sum_{\sigma\in S_n} wt(\sigma) = \sum_{i=1}^{n}\sum_{\substack{\sigma\in S_n\\ \sigma(n)=i}} wt(\sigma) = \\ &= \left(\prod_{j=2}^{n}\xi_j\right) P_{n-1}(q;\xi;q_3,\ldots,q_n;\xi_3,\ldots,\xi_n) + \sum_{i=2}^{n}\left(\prod_{j=i+1}^{n}\xi_j\right)\times \\ &\times P_{n-1}(q;\xi;qq_2,\ldots,qq_{i-1},q_{i+1},\ldots,q_n;\xi\xi_2,\ldots,\xi\xi_{i-1},\xi_{i+1},\ldots,\xi_n) \end{aligned}$$

7. Our vector of occurrences is $\mathbf{0}$, so we don't have to take any derivatives here.

8. Let $q = \xi = 1$, all $q_j = \xi_j = 0$, and substitute into the above formula.

Let

$$P(n, I) = P_n(0; 0; \underbrace{0, \dots, 0}_{I-1}, \underbrace{1, \dots, 1}_{n-I}; \underbrace{0, \dots, 0}_{I-1}, \underbrace{1, \dots, 1}_{n-I}),$$

then we have

$$P(n, I) = \sum_{i=I}^{n} P(n-1, i-1)$$

for $n \geq 2$ and $1 \leq I \leq n$ if we let $P(n, 0) = P(n, 1)$. Clearly, we need to find $P(n, 1)$, i.e. when we have no additional restrictions by auxiliary subpatterns. Notice that $P(n, I)$ is the number of permutations which avoid 123 and 213 and have no subsequences $i_1 i_2$ such that either $i_1 < i_2 \leq I$ or $i_2 < i_1 \leq I$. Letting $i_1 = 1$, $i_2 = 2$, we see that this is impossible for $I \geq 2$, so $P(n, I) = 0$ for $I > 1$. Hence,

$$P(n, 1) = 2P(n-1, 1).$$

Together with the fact that $P(1, 1) = 1$, this implies that

$$P(n, 1) = 2^{n-1}$$

as in Table 1.3.

Remark 1.6 One should note that Noonan-Zeilberger method as described in the example above works especially well for single patterns in the symmetry classes of $12\dots\ell$ and $12\dots(\ell-2)\ell(\ell-1)$. We use a close but somewhat different approach for sets of two and more patterns of length three.

1.7 West equivalence

In his thesis, West [27] not only proved that patterns 1234, 1243 and 2143 are Wilf-equivalent, but also established a much more general fact.

Theorem 1.21 (West) *Let $A = a_3 \dots a_\ell$ be a permutation of $3\dots\ell$. Then $12A \sim 21A$, i.e. patterns $12A$ and $21A$ are Wilf-equivalent.*

Here $12A$ and $21A$ are permutations which arise by concatenation of 12 or 21 and A.

Clearly, this theorem implies that $123 \sim 132$ as well as Theorem 1.9 above (since reversal and complementation also imply Wilf-equivalence).

Based on his data for small patterns and permutations, West made the following

Conjecture 1.4 (West) *Let $I_t = 12\dots t$, $J_t = t(t-1)\dots 1$, and let $A = a_{t+1}\dots a_\ell$ be any permutation of $(t+1)\dots\ell$. Then $I_tA \sim J_tA$.*

Again, I_tA and J_tA are concatenations of I_t or J_t and A.

Note that Theorem 1.21 is a special case of Conjecture 1.4 (which we will call West's Conjecture) with $t = 2$. Later, the conjecture was proved for $t = 3$ by Babson and West [30]. Finally, West thought to have proved the general case of the this conjecture in [31]. However, his current proof contains an incorrect generalization of the proof of a technical sublemma (called the sliding lemma) which was used in [30] as well. However, this proof of the sliding lemma, although incorrect in general, still works for $t = 4$, and hence, West's Conjecture holds for $t = 4$ as well.

Interestingly, this sliding lemma is sufficient, but may not be necessary for West's main lemma to hold. Hence, it may be possible to prove West's Conjecture without the use of the sliding lemma.

We discuss West's method in detail in Chapter 9 and follow it to prove analogs on words for his results on permutations and make further conjectures regarding cardinalities of Wilf classes on words and on permutations.

For $t = 2, \ldots, \ell - 1$, we define a set of functions $w_t : S_\ell \to S_\ell$ as follows: given a pattern P,

$$w_t(P) = \begin{cases} J_t A, & \text{if } P = I_t A, \\ I_t A, & \text{if } P = J_t A, \\ P, & \text{otherwise,} \end{cases} \tag{1.11}$$

where A is any permutation of $(t+1, \ldots, \ell)$.

Definition 1.9 *We call two patterns $\tau_1, \tau_2 \in S_\ell$ West-equivalent on words (resp. on permutations) and say that they belong to the same West class on words (resp. on permutations) if they are related by the action of the group $H_w = \langle r, c, w_2, \ldots, w_{\ell-1} \rangle$ (resp. $H_p = \langle r, c, i, w_2, \ldots, w_{\ell-1} \rangle$) on S_ℓ.*

Example 1.7 Patterns 1234, 1243, 1432 and 2143 are West-equivalent, for example, via the following path:

$$\mathbf{1432} \overset{r}{\leftrightarrow} 2341 \overset{c}{\leftrightarrow} 3214 \overset{w_3}{\leftrightarrow} \mathbf{1234} \overset{w_2}{\leftrightarrow} 2134 \overset{r}{\leftrightarrow} 4312 \overset{c}{\leftrightarrow} \mathbf{1243} \overset{w_2}{\leftrightarrow} \mathbf{2143}.$$

Remark 1.7 Note that w_1 is the identity, and w_ℓ is a special case of reversal r, hence, they need not be defined.

Notation 1.4 We denote the West class of τ by $\widetilde{\tau}$ (i.e. $\widetilde{\tau}^w$ on words and $\widetilde{\tau}^p$ on permutations).

Example 1.8 In S_4, there are 5 West classes on words ($\widetilde{1234}$, $\widetilde{1324}$, $\widetilde{1342}$, $\widetilde{1423}$, $\widetilde{2413}$) and 4 West classes on permutations ($\widetilde{1234}$, $\widetilde{1324}$, $\widetilde{1342}$, $\widetilde{2413}$) since $1423 = 1342^{-1}$.

Cardinality class $\boxed{T}$	Symmetry class $\overline{T}$	Cardinality $\lvert S_n(T)\rvert$
$\boxed{123}$	$\overline{123}$, $\overline{132}$	$c_n = \frac{1}{n+1}\binom{2n}{n}$
$\boxed{123,\ 132}$	$\overline{\{123, 132\}}$ $\overline{\{132, 213\}}$ $\overline{\{132, 231\}}$ $\overline{\{132, 312\}}$	2^{n-1}
$\boxed{123,\ 231}$	$\overline{\{123, 231\}}$	$\binom{n}{2} + 1$
$\boxed{123,\ 321}$	$\overline{\{123, 321\}}$	0 if $n \geq 5$
$\boxed{123,\ 132,\ 213}$	$\overline{\{123, 132, 213\}}$	F_{n+1}
$\boxed{123,\ 132,\ 231}$	$\overline{\{123, 132, 231\}}$ $\overline{\{132, 213, 231\}}$ $\overline{\{123, 213, 231\}}$ $\overline{\{123, 231, 312\}}$	n
$\boxed{123,\ 321,\ 213}$	$\overline{\{123, 321, 213\}}$	0 if $n \geq 5$
$\boxed{T}$, $\lvert T\rvert = 4, 5$, $T \supset \{123, 321\}$	$\cdots$	0 if $n \geq 5$
$\boxed{T}$, $\lvert T\rvert = 4, 5$, $T \not\supset \{123, 321\}$	$\cdots$	2 if $n \geq 2$
$\boxed{S_3}$	$\overline{S_3}$	0 if $n \geq 3$

Table 1.3: Wilf classes for subsets of S_3 on permutations

Chapter 2

Overview of new results

In this chapter, we list the major results of our thesis contained in the following chapters. First of all, in this thesis we only consider patterns which are permutations, leaving patterns with repeated letters for future works.

2.1 Forbidden sets of patterns in S_3

In Part II, we look at all forbidden sets of 3-letter permutation patterns. Our main result for single 3-letter patterns is the following (see Theorem 3.1 on page 56, and Theorems 3.2 and 3.3 on page 59).

THEOREM 2.1 *For any pattern $\tau \in S_3$ and any $k \geq 2$, $n \geq 0$,*

$$|[k]^n(\tau)| = 2^{n-2(k-2)} \sum_{j=0}^{k-2} a_{k-2,j} \binom{n+2j}{2j}, \tag{2.1}$$

where

$$a_{k,j} = \sum_{i=j}^{k} c_i d_{k-i}, \quad c_j = \frac{1}{j+1}\binom{2j}{j}, \quad d_j = \binom{2j}{j}. \tag{2.2}$$

Thus, all patterns in S_3 constitute a single Wilf class on words as they do on permutations.

REMARK 2.1 Note that $a_{k,0} = \sum_{j=0}^{k} c_j d_{k-j} = \frac{1}{2} d_{k+1}$, $a_{k,k} = c_k$.

REMARK 2.2 Also note that $2 = \ell - 1$ for $\ell = 3$ (cf. Theorem 1.7).

EXAMPLE 2.1 Let $k = 2$, then $k - 2 = 0$, $a_{0,0} = 1$, and (2.1) becomes the obvious $|[2]^n(\tau)| = 2^n$ for any pattern $\tau \in S_3$. This is, in fact, true for any pattern τ with at least 3 distinct letters (see Remark 1.2).

EXAMPLE 2.2 Let $k = 3$, then $k - 2 = 1$, $a_{1,0} = 3$, $a_{1,1} = 1$, and (2.1) becomes $|[3]^n(\tau)| = 2^{n-2}\left(\binom{n+2}{2} + 3\right)$ for any pattern $\tau \in S_3$. Thus, for example, $|[3]^3(\tau)| = 2 \cdot (10 + 3) = 26 = 3^3 - 1$ as expected.

For other subsets of S_3 (containing more than one pattern), we obtain the following results. We find the number of Wilf classes of subsets of S_3 according to their cardinalities (see Table 2.1) as well as the classes themselves and the cardinalities of sets of words avoiding them (see Table 2.3 on page 40). For comparison, the corresponding data on permutations (as found in Simion, Schmidt [21]) is given in Table 2.2 on this page and in Table 1.3 on page 34.

Cardinality of set of patterns	1	2	3	4	5	6
Number of symmetry classes	2	6	6	6	2	1
Number of Wilf classes	1	5	5	6	2	1

Table 2.1: Number of Wilf classes on words for subsets in S_3 (new results)

Cardinality of set of patterns	1	2	3	4	5	6
Number of symmetry classes	2	5	5	5	2	1
Number of Wilf classes	1	3	3	3	2	1

Table 2.2: Number of Wilf classes on permutations for subsets of S_3 (see [21])

We also obtain the following theorem as a corollary of Theorem 2.1 (see Theorem 8.1 on page 115).

THEOREM 2.2 $\lim_{n\to\infty} |[k]^n(T)|^{\frac{1}{n}} = 2$ *for any* $T \subseteq S_3$ *and* $k \geq 2$.

2.2 Single patterns in S_ℓ

In Part III, we consider single permutation patterns of length ℓ and try to determine equivalence relations among them which imply Wilf equivalence.

THEOREM 2.3 *For any $\tau_1, \tau_2 \in S_\ell$, if $\tau_1 = I_t A$, $\tau = J_t A$, $t \leq 4$, then $\tau_1 \sim \tau_2$ on words (as well as on permutations).*

As a corollary, we have

THEOREM 2.4 *West equivalence implies Wilf equivalence on S_ℓ for $\ell \leq 5$ on words (as well as on permutations).*

We also reduce the general case of the above theorem (i.e. any $t \in \mathbb{N}$) to the proof of the sliding lemma. Had it not been for the unfortunate fault in its current proof by West, which this author discovered late in the writing of this thesis, we would have been able to state the above theorems without the restrictions $t \leq 4$ and $\ell \leq 5$.

Based on our numerical data for $\ell \leq 5$, we can conclude that the converse of Theorem 2.4 is also true.

THEOREM 2.5 *On words, Wilf classes of S_ℓ are exactly West classes of S_ℓ for $\ell \leq 5$.*

This leads us to conjecture that, in general, Wilf classes of single permutation patterns on words are exactly their West classes.

Cardinality class $\boxed{T}$	Symmetry class $\overline{T}$	Cardinality $\lvert[k]^n(T)\rvert$
$\boxed{123}$	$\overline{123}$, $\overline{132}$	eq. 2.1 on page 36
$\boxed{123,\ 132}$	$\overline{\{123, 132\}}$	eq. 4.11 on page 66
	$\overline{\{132, 312\}}$	eq. 4.24 on page 71
$\boxed{123,\ 321}$	$\overline{\{123, 321\}}$	eq. 4.19 on page 68
$\boxed{132,\ 231}$	$\overline{\{132, 231\}}$	eq. 4.31 on page 73
$\boxed{132,\ 213}$	$\overline{\{132, 213\}}$	eq. 4.36 on page 76
$\boxed{123,\ 231}$	$\overline{\{123, 231\}}$	eq. 4.42 on page 78
$\boxed{123,\ 132,\ 213}$	$\overline{\{123, 132, 213\}}$	eq. 5.6 on page 83
$\boxed{123,\ 213,\ 231}$	$\overline{\{123, 213, 231\}}$	eq. 5.14 on page 86
$\boxed{123,\ 321,\ 213}$	$\overline{\{123, 321, 213\}}$	eq. 5.23 on page 89
$\boxed{123,\ 132,\ 231}$	$\overline{\{123, 132, 231\}}$	eq. 5.29 on page 92
	$\overline{\{132, 213, 231\}}$	
$\boxed{123,\ 231,\ 312}$	$\overline{\{123, 231, 312\}}$	eq. 5.36 on page 96
$\boxed{123,\ 132,\ 213,\ 231}$	$\overline{\{123, 132, 213, 231\}}$	eq. 6.6 on page 101
$\boxed{123,\ 132,\ 231,\ 312}$	$\overline{\{123, 132, 231, 312\}}$	eq. 6.11 on page 103
$\boxed{123,\ 132,\ 213,\ 321}$	$\overline{\{123, 132, 213, 321\}}$	eq. 6.16 on page 104
$\boxed{123,\ 132,\ 231,\ 321}$	$\overline{\{123, 132, 231, 321\}}$	eq. 6.21 on page 106
$\boxed{123,\ 213,\ 231,\ 321}$	$\overline{\{123, 213, 231, 321\}}$	eq. 6.24 on page 107
$\boxed{132,\ 213,\ 231,\ 312}$	$\overline{\{132, 213, 231, 312\}}$	eq. 6.26 on page 109
$\boxed{S_3 \setminus \{123\}}$	$\overline{S_3 \setminus \{123\}}$	eq. 7.1 on page 111
$\boxed{S_3 \setminus \{132\}}$	$\overline{S_3 \setminus \{132\}}$	eq. 7.2 on page 112
$\boxed{S_3}$	$\overline{S_3}$	eq. 7.3 on page 113

Table 2.3: Wilf classes for subsets of S_3 on words (new results)

Part II

Sets of forbidden 3-letter patterns

Chapter 3

Single forbidden 3-letter patterns

We would like to start by finding the number of n-letter words σ over a totally ordered k-letter alphabet, $\{1, \ldots, k\}$, which avoid a given pattern in S_3 (in this thesis we will only consider patterns which are permutations). To do this, we only need to investigate two patterns, 123 and 312, since the reversal map, $\sigma(i) \to \sigma(n+1-i)$, and the complementation map, $\sigma(i) \to k+1-\sigma(i)$, subdivide the patterns in S_3 into 2 equivalence classes with respect to pattern avoidance: $\{123, 321\}$ and $\{132, 213, 231, 312\}$.

3.1 Forbidden pattern 123

We start with finding the number of n-letter words over a k-letter alphabet, $\{1,\ldots,k\}$, which avoid a 123 pattern. To do that, we use the Noonan-Zeilberger method [14]. We also borrow the notation from there, with only slight changes.

Definition 3.1 *For any n-letter word σ, let $\varphi(\sigma)$ be the number of occurrences of the 123 pattern in σ, and for $j = 2,\ldots,k$, let $\varphi_j(\sigma)$ be the number of strict noninversions $i_1 i_2$ (occurrences of the 12 pattern) such that $i_1 < i_2 \leq j$. Define the weight of σ as*

$$wt(\sigma) = q^{\varphi(\sigma)} \prod_{j=2}^{k} q_j^{\varphi_j(\sigma)}.$$

Finally, let

$$P_n(q; q_2,\ldots,q_k) = \sum_{\sigma\in[k]^n} wt(\sigma),$$

where $[k] = \{1,\ldots,k\}$.

Clearly, what we want to find is $P_n(0; 1,\ldots,1)$. First, we have to find the recurrence relation for $P_n(q; q_2,\ldots,q_k)$.

Definition 3.2 *For each word σ, let $s(\sigma) = (m_1,\ldots,m_k)$, where m_i is the number of occurrences of the letter i in σ (so $m_1 + \ldots + m_k = n$). We call*

$s(\sigma)$ *the supply vector of* σ, and $m_i = s_i(\sigma)$, *the supply of* i. *Let*

$$P_s(q; q_2, \dots, q_k) = \sum_{\{\sigma \in [k]^n | s(\sigma) = s\}} wt(\sigma)$$

Let σ be any n-letter word such that $\sigma(n) = i$, and let the word σ_1 be obtained by deleting the last letter from σ. Then, for $i = 1$, $wt(\sigma) = wt(\sigma_1)$, otherwise,

$$\begin{aligned} wt(\sigma) &= wt(\sigma_1) q_i^{\sum_{j=1}^{i-1} m_j} q^{\sum_{j=2}^{i-1} \varphi_j(\sigma)} \\ &= q^{\varphi(\sigma_1)} q_i^{\sum_{j=1}^{i-1} m_j} \prod_{j=1}^{i-1} (qq_j)^{\varphi_j(\sigma_1)} \prod_{j=i}^{k} q_j^{\varphi_j(\sigma_1)}. \end{aligned}$$

Summing over all σ with a given supply vector s, we get

$$\begin{aligned} P_s(q; q_2, \dots, q_k) = \delta_{s-e_1 \geq 0} P_{s-e_1}(q; q_2, \dots, q_k) + \\ + \sum_{i=2}^{k} q_i^{\sum_{j=1}^{i-1} m_j} \delta_{s-e_i \geq 0} P_{s-e_i}(q; qq_2, \dots, qq_{i-1}, q_i, \dots, q_k), \end{aligned}$$

where e_i is the ith unit supply vector, 0=(0,...,0), δ denotes the truth value of the corresponding proposition, and inequalities hold componentwise.

Let

$$P_s(k, I) = P_s(0; \underbrace{0, \dots, 0}_{I-1}, \underbrace{1, \dots, 1}_{k-I})$$

be the number of words with the given supply vector s which contain no occurences of 123 pattern and no strict noninversions $i_1 i_2$ such that $i_1 < i_2 \leq I$. Also, let J_s be the smallest letter which occurs in s (i.e. $m_i = 0$ for $i < J_s$, and $m_{J_s} > 0$). Then, for $I = 1, \ldots, k$, substituting $q = q_2 = \ldots = q_I = 0$, $q_{I+1} = \ldots = q_k = 1$ into the recurrence relation above, we get

$$\begin{aligned} P_s(k, I) = &\sum_{i=1}^{I} \delta_{s-e_i \geq 0} P_{s-e_i}(k, I) \delta(m_1 = \ldots = m_{i-1} = 0) + \\ &+ \sum_{i=I+1}^{k} \delta_{s-e_i \geq 0} P_{s-e_i}(k, i-1). \end{aligned}$$

We see that if $J_s > I$, then the first sum on the right-hand side of the equation is 0, since $m_i = 0$, $i = 1, \ldots, I$, implies that $\delta_{s-e_i \geq 0} = 0$ for all $i \leq I$. Let us examine the summands of this sum if $J_s \leq I$. If $i < J_s$, then $m_i = 0$, so $\delta_{s-e_i \geq 0} = 0$. If $i > J_s$, then $\delta(m_1 = \ldots = m_{i-1} = 0) = 0$, since $m_{J_s} > 0$. Thus, the only possibly nonzero summand occurs when $i = J_s$, so our recursive relation takes the form

$$P_s(k, I) = \delta_{J_s \leq I} P_{s-e_{J_s}}(k, I) + \sum_{i=I+1}^{k} \delta_{s-e_i \geq 0} P_{s-e_i}(k, i-1). \tag{3.1}$$

Summing this over all $s \in \mathbb{N}^k$ such that $|s| = m_1 + \ldots + m_k = n$, we get

$$\begin{aligned} P_n(k, I) &= \sum_{i=1}^{I} \sum_{\{s \,:\, |s|=n, J_s=i\}} P_{s-e_i}(k, I) + \sum_{i=I}^{k-1} P_{n-1}(k, i) = \\ &= \sum_{i=1}^{I} P_{n-1}(k-i+1, I-i+1) + \sum_{i=I}^{k-1} P_{n-1}(k, i) \end{aligned} \tag{3.2}$$

Therefore, for $I > 1$,

$$P_n(k+1, I) + P_n(k, I) - P_n(k, I-1) - P_n(k+1, I+1) =$$

$$= 2P_{n-1}(k+1, I) - P_{n-1}(k, I-1) - P_{n-1}(k+1, I+1), \quad (3.3)$$

and for $I = 1$,

$$P_n(k+1, 1) - P_n(k+1, 2) = 2P_{n-1}(k+1, 1) - P_{n-1}(k, 1) - P_{n-1}(k+1, 2). \quad (3.4)$$

Let $h_I(n, k) = P_n(k, k-I)$, then from (3.3) and (3.4), we easily get for $k > I+1, I \geq 1$,

$$h_{I+1}(n, k+1) - 2h_{I+1}(n-1, k+1) - h_{I+1}(n, k) + h_{I+1}(n-1, k) =$$

$$= h_I(n, k+1) - h_I(n-1, k+1) - h_I(n, k) \quad (3.5)$$

and, for $k = I+1, I \geq 1$,

$$h_{I+1}(n, I+2) - 2h_{I+1}(n-1, I+2) =$$

$$= h_I(n, I+2) - h_I(n-1, I+2) - h_I(n-1, I+1). \quad (3.6)$$

Let $h_{I,k}(x) = \sum_{n\geq 0} h_I(n, k)x^n$, then summing (3.5) and (3.6) over n and considering that $h_I(0, k) = 1$ for all I, k such that $k \geq I+1$, we get, for $I \geq 1$,

$$(1-2x)h_{I+1,k+1}(x) - (1-x)h_{I+1,k}(x) =$$

$$= (1-x)h_{I,k+1}(x) - h_{I,k}(x), \quad k > I+1, \quad (3.7)$$

$$(1-2x)h_{I+1,I+2}(x) = (1-x)h_{I,I+2}(x) - xh_{I,I+1}(x) \tag{3.8}$$

Let $H_I(x,y) = \sum_{k=I+1}^{\infty} h_{I,k}(x)y^k$, $u_I(x) = h_{I,I+1}(x)$, then it easily follows from (3.7) and (3.8) that

$$\begin{aligned}((1-2x)-(1-x)y)H_{I+1}(x,y) - (1-x-y)H_I(x,y) = \\ = (1-x)u_I(x)(y-1)y^{I+1} = \psi_I(x,y), \quad I \geq 1\end{aligned} \tag{3.9}$$

from which we get that

$$\begin{aligned}H_I(x,y) = H_1(x,y)\left(\frac{1-x-y}{((1-2x)-(1-x)y}\right)^{I-1} + \\ + \sum_{j=1}^{I-1} \frac{(1-x-y)^{I-1-j}}{((1-2x)-(1-x)y)^{I-j}}\psi_j(x,y)\end{aligned} \tag{3.10}$$

To find $H_1(x,y)$, we note that $h_1(n,k) = P_n(k,k-1)$ is the number of n-letter words which contain no 123 patterns and whose only strict noninversions are those containing the letter k, so we get all such words by taking a nonincreasing (possibly empty) sequence of letters $\{1,\ldots,k-1\}$ of length $\leq n$ and adding letters k with no restriction on position to get a word of length n. Therefore, it's easy to see that

$$h_1(n,k) = \sum_{m=0}^{n} \binom{n}{m}\binom{m+k-2}{k-2}, \tag{3.11}$$

so

$$H_1(x,y) = \frac{y^2}{(1-2x)-(1-x)y}. \tag{3.12}$$

Substituting this into (3.10), we get that

$$
\begin{aligned}
H_I(x,y) = {} & \frac{(1-x-y)^{I-1}}{((1-2x)-(1-x)y)^I} y^2 + \\
& + \sum_{j=1}^{I-1} \frac{(1-x-y)^{I-1-j}}{((1-2x)-(1-x)y)^{I-j}} (1-x) u_j(x)(y-1) y^{j+1}
\end{aligned}
\tag{3.13}
$$

How does this relate to what we want to find, i.e. $P_n(1,\ldots,1;0)$? We have $P_n(1,\ldots,1;0) = P_n(k,1) = h_{k-1}(n,k)$, so the generating function $u_I(x) = \sum_{n=0}^{\infty} h_I(n, I+1) x^n$ is the one for the number of words over $(I+1)$-letter alphabet with no 123 patterns and no restrictions on 12 patterns, which is exactly what we want. On the right-hand side of (3.13) we only have $u_1(x), \ldots, u_{I-1}(x)$, so if we know these, we can find out $u_I(x) = \langle y^{I+1} \rangle H_I(x,y)$ (i.e. the coefficient of y^{I+1} in $H_I(x,y)$).

To find $\langle y^{I+1} \rangle H_I(x,y)$, we first find

$$
f_{l,m}(x) = \langle y^l \rangle \left(\frac{(1-x-y)^m}{((1-2x)-(1-x)y)^{m+1}} \right), \tag{3.14}
$$

where $l \geq 0$, $m \geq 0$. Since

$$
\frac{1-x-y}{(1-2x)-(1-x)y} = \frac{1}{1-x} \left(1 + \frac{x^2}{(1-2x)-(1-x)y} \right), \tag{3.15}
$$

we eventually find out through some technical manipulations that

$$
f_{l,m}(x) = \frac{(1-x)^{l-m}}{(1-2x)^{l+1}} \sum_k \binom{l+k}{k} \binom{m}{k} \left(\frac{x^2}{1-2x} \right)^k \tag{3.16}
$$

Taking coefficients of y^{I+1} on both sides of (3.13) and substituting (3.16), we obtain the following recursive relation for $u_I(x)$:

$$\begin{aligned} u_I(x) = &\frac{1}{(1-2x)^I}\sum_k \binom{I-1+k}{k}\binom{I-1}{k}\left(\frac{x^2}{1-2x}\right)^k \\ &+\sum_{j=1}^{I-1} u_j(x)\frac{1-x}{(1-2x)^{I-j}}\sum_k \binom{I-j-1+k}{k}\binom{I-j-1}{k}\left(\frac{x^2}{1-2x}\right)^k \\ &-\sum_{j=1}^{I-1} u_j(x)\frac{(1-x)^2}{(1-2x)^{I-j+1}}\sum_k \binom{I-j+k}{k}\binom{I-j-1}{k}\left(\frac{x^2}{1-2x}\right)^k \end{aligned} \tag{3.17}$$

Since we know

$$u_1(x) = \langle y^2\rangle H_1(x,y) = \frac{1}{1-2x},$$

we can find all successive $u_I(x)$'s. Here are the first three of them:

$$\begin{aligned} u_1(x) &= \frac{1}{1-2x}, \\ u_2(x) &= \frac{1}{4}\left(\frac{1}{(1-2x)^3}+\frac{3}{1-2x}\right), \\ u_3(x) &= \frac{1}{8}\left(\frac{1}{(1-2x)^5}+\frac{2}{(1-2x)^3}+\frac{5}{1-2x}\right), \end{aligned}$$

from which we conclude that

$$\begin{aligned} u_{1,n} &= 2^n, \\ u_{2,n} &= \left(\binom{n+2}{2}+3\right)2^{n-2}, \end{aligned}$$

$$u_{3,n} = \left(\binom{n+4}{4} + 2\binom{n+2}{2} + 5 \right) 2^{n-3},$$

where $u_{I,n} = \langle x^n \rangle u_I(x)$ is the number of n-letter words over $(I+1)$-letter alphabet with no 123 patterns. Finding several more successive $u_I(x)$'s, we see that they have the form

$$u_I(x) = \frac{1}{4^{I-1}} \sum_{m=0}^{I-1} a_{I,m} \frac{1}{(1-2x)^{2m+1}}, \tag{3.18}$$

so

$$u_{I,n} = \langle x^n \rangle u_I(x) = 2^{n-2(I-1)} \sum_{m=0}^{I-1} a_{I,m} \binom{n+2m}{2m}. \tag{3.19}$$

Arranging the $a_{I,m}$'s in the form of a triangle with $a_{I,m}$ being $(m+1)$-st from the right in the I-th row, we have

$$\begin{array}{c}
1 \\
1 \quad 3 \\
2 \quad 4 \quad 10 \\
5 \quad 9 \quad 15 \quad 35 \\
14 \quad 24 \quad 36 \quad 56 \quad 126 \\
42 \quad 70 \quad 100 \quad 140 \quad 210 \quad 462 \\
132 \quad 216 \quad 300 \quad 400 \quad 540 \quad 792 \quad 1716
\end{array}$$

This allows us to conjecture that

$$a_{I,m} = \frac{1}{2}\binom{2I}{I} - \sum_{j=0}^{m-1} \frac{1}{j+1}\binom{2j}{j}\binom{2(I-1-j)}{I-1-j}, \tag{3.20}$$

and indeed, substituting the resulting $u_I(x)$'s into (3.17), we see that it holds for any $I \geq 1$.

However, the above solution involves a rather complicated guess and just as complicated a verification of the identity resulting from the substitution, so here is another way of arriving at (3.18) and (3.20).

For $k \geq 0$, let

$$\begin{aligned}
v_k(x) &= u_{k+1}(x), \\
p_k(x) &= \frac{1}{(1-2x)^{k+1}} \sum_i \binom{k+i}{i}\binom{k}{i}\left(\frac{x^2}{1-2x}\right)^i, \\
q_k(x) &= \frac{1}{(1-2x)^{k+2}} \sum_i \binom{k+1+i}{i}\binom{k}{i}\left(\frac{x^2}{1-2x}\right)^i, \\
V(x,y) &= \sum_{k=0}^{\infty} v_k(x) y^k, \\
P(x,y) &= \sum_{k=0}^{\infty} p_k(x) y^k = \frac{1}{1-2x} \sum_{i,k\geq 0} \binom{k+i}{i}\binom{k}{i}\left(\frac{x^2}{1-2x}\right)^i \left(\frac{y}{1-2x}\right)^k, \\
Q(x,y) &= \sum_{k=0}^{\infty} q_k(x) y^k \\
&= \frac{1}{(1-2x)^2} \sum_{i,k\geq 0} \binom{k+1+i}{i}\binom{k}{i}\left(\frac{x^2}{1-2x}\right)^i \left(\frac{y}{1-2x}\right)^k.
\end{aligned}$$

Then

$$v_k(x) = p_k(x) + (1-x)\sum_{j=0}^{k-1} v_j(x)p_{k-1-j}(x) - (1-x)^2\sum_{j=0}^{k-1} v_j(x)q_{k-1-j}(x),$$

so summing over $k \geq 0$, we get

$$V(x,y) = P(x,y) + y(1-x)V(x,y)P(x,y) - y(1-x)^2V(x,y)Q(x,y),$$

which implies that

$$V(x,y) = \frac{P(x,y)}{1-(1-x)yP(x,y)+(1-x)^2yQ(x,y)}. \tag{3.21}$$

We also note that

$$U(x,y) = \sum_{k=0}^{\infty} u_k(x)y^k = \frac{1}{1-x} + yV(x,y)$$

and that we must prove that

$$V(x,y) = \sum_{k=0}^{\infty}\sum_{m=0}^{k} a_{k+1,m}\frac{1}{(1-2x)^{2m+1}}y^k.$$

Let

$$F(s,t) = \sum_{i,k\geq 0}\binom{i+k}{k}\binom{k}{i}s^it^k,$$

$$G(s,t) = \sum_{i,k\geq 0}\binom{i+k+1}{k+1}\binom{k}{i}s^it^k,$$

then

$$P(x,y) = \frac{1}{1-2x}F\left(\frac{x^2}{1-2x},\frac{y}{1-2x}\right),$$

$$Q(x,y) = \frac{1}{(1-2x)^2} G\left(\frac{x^2}{1-2x}, \frac{y}{1-2x}\right).$$

Since $\binom{i+k}{k}\binom{k}{i} = \binom{i+k}{i,i,k-i} = \binom{i+k}{2i}\binom{2i}{i} = \binom{(k-i)+2i}{2i}\binom{2i}{i}$, it is easy to prove that

$$F(s,t) = \frac{1}{\sqrt{(1-t)^2 - 4st}} = \frac{1}{\sqrt{(t-(2s+1))^2 - 4s(s+1)}},$$

which implies that

$$P(x,y) = \frac{1}{\sqrt{(1-2x-y)^2 - 4x^2y}} = \frac{1}{\sqrt{(1-y)((1-2x)^2 - y)}}.$$

We also have that $\binom{i+k+1}{k+1} = (1 + \frac{i}{k+1})\binom{i+k}{k}$, which implies that

$$\begin{aligned} G(s,t) &= F(s,t) + \frac{s}{t}\left(\frac{\partial}{\partial s}\int_0^t F(s,r)dr\right) = \\ &= \frac{1 - t - \sqrt{(1-t)^2 - 4st}}{t\sqrt{(1-t)^2 - 4st}\left(1 + 2s - t - \sqrt{(1-t)^2 - 4st}\right)}. \end{aligned}$$

Multiplying both the numerator and the denominator of $G(s,t)$ by $1 + 2s - t + \sqrt{(1-t)^2 - 4st}$ and cancelling the factor of $2s$ in both places, we get

$$G(s,t) = \frac{1 + t - \sqrt{(1-t)^2 - 4st}}{2t(s+1)\sqrt{(1-t)^2 - 4st}}.$$

Since $\frac{x^2}{1-2x} + 1 = \frac{(1-x)^2}{1-2x}$, this implies that

$$Q(x,y) = \frac{1 - 2x + y - \sqrt{(1-y)((1-2x)^2 - y)}}{y(1-x)^2\sqrt{(1-y)((1-2x)^2 - y)}}.$$

Finally, we substitute $P(x,y)$ and $Q(x,y)$ from above into (3.21) and multiply the numerator and denominator of $V(x,y)$ by $\sqrt{(1-y)((1-2x)^2 - y)}$

to find that

$$\begin{aligned} V(x,y) &= \frac{1}{\sqrt{(1-y)((1-2x)^2-y)} - (1-x)y + \frac{1-2x+y-\sqrt{(1-y)((1-2x)^2-y)}}{2}} = \\ &= \frac{2}{(1-2x)(1-y) + \sqrt{(1-y)((1-2x)^2-y)}}. \end{aligned} \tag{3.22}$$

Let $\tilde{V}(z,y)$ be such that $V(x,y) = \tilde{V}(\frac{1}{1-2x}, y)$ (in other words, $\tilde{V}(z,y) = V(\frac{z-1}{2z}, y))$, then letting $C_j = \frac{1}{j+1}\binom{2j}{j}$, we have

$$\begin{aligned} \tilde{V}(z,y) &= \frac{2z}{(1-y) + \sqrt{(1-y)(1-yz^2)}} = \\ &= \frac{2z\left(\sqrt{(1-y)(1-yz^2)} - (1-y)\right)}{(z^2-1)y(1-y)} = \\ &= \frac{2z}{1-z^2}\,\frac{1}{y}\left(\sqrt{1-yz^2}\frac{1}{\sqrt{1-y}} - 1\right) = \\ &= \frac{2z}{1-z^2}\,\frac{1}{y}\left(\left(1 - 2\sum_{k=0}^{\infty} C_k \frac{(yz^2)^{k+1}}{4^k}\right)\left(\sum_{k=0}^{\infty}\binom{2k}{k}\frac{y^k}{4^k}\right) - 1\right) = \\ &= \frac{2z}{1-z^2}\sum_{k=1}^{\infty}\left(\binom{2k}{k} - 2\sum_{m=1}^{k} C_{m-1}\binom{2(k-m)}{k-m}z^{2m}\right)\frac{y^{k-1}}{4^k} = \\ &= \frac{z}{1-z^2}\sum_{k=0}^{\infty}\left(\frac{1}{2}\binom{2(k+1)}{k+1} - \sum_{m=0}^{k} C_m\binom{2(k-m)}{k-m}z^{2m+2}\right)\frac{y^k}{4^k} = \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k}\left(\frac{1}{2}\binom{2(k+1)}{k+1} - \sum_{j=0}^{m-1} C_j\binom{2(k-j)}{k-j}\right)z^{2m+1}\frac{y^k}{4^k} = \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} a_{k+1,m} z^{2m+1} y^k, \end{aligned} \tag{3.23}$$

which is exactly what we wanted to prove.

We will explain the passage from the third line from the bottom of (3.23) to the next line. Note that

$$\frac{1}{2}\binom{2(k+1)}{k+1} = \sum_{m=0}^{k} C_m \binom{2(k-m)}{k-m},$$

which can be easily derived by comparing the generating functions of both sides of the equation. This implies that

$$f(t) = \frac{1}{2}\binom{2(k+1)}{k+1} - \sum_{m=0}^{k} C_m \binom{2(k-m)}{k-m} t^{m+1}$$

divides $1-t$, so $g(t) = \frac{f(t)}{1-t}$ is a polynomial of degree k, and

$$\langle t^m \rangle g(t) = \sum_{j=0}^{m} \langle t^j \rangle f(t), \quad m = 0, 1, \ldots, k.$$

Thus, we have the following

Theorem 3.1 *The number of n-letter words over k-letter alphabet which do not contain the 123 pattern is equal to 1 if $k=1$, and*

$$u_{k-1,n} = 2^{n-2(k-2)} \sum_{m=0}^{k-2} \left(\frac{1}{2}\binom{2(k-1)}{k-1} - \sum_{j=0}^{m-1} C_j \binom{2(k-2-j)}{k-2-j} \right) \binom{n+2m}{n}$$

if $k \geq 2$.

3.2 Forbidden pattern 312

Let us now find the number of permutations which avoid the 312 pattern. Here and further, we will regard the definitions φ, φ_j, wt, P_τ and P_n as local to the current problem.

Definition 3.3 *For any $\sigma \in [k]^n$, let $\varphi(\sigma)$ be the number of occurrences of the 312 pattern in σ, and let $\varphi_j(\sigma)$ $(j = 2, \ldots, k)$, $wt(\sigma)$, $P_s(q; q_2, \ldots, q_k)$ and $P_n(q; q_2, \ldots, q_k)$ be defined as in Definitions 3.1 and 3.2.*

Again, to find is $P_n(0; 1, \ldots, 1)$ (which is what we want), we have to start with finding the recurrence relation for $P_s(q; q_2, \ldots, q_k)$.

In this case, given an n-letter word σ, we let σ_1 be the word obtained by deleting the first letter of σ. Suppose that $\sigma(1) = i$, then

$$wt(\sigma) = wt(\sigma_1) \left(\prod_{j=i+1}^{k} q_j^{m_j} \right) q^{\sum_{j=2}^{i-1} \varphi_j(\sigma)} =$$

$$= q^{\varphi(\sigma_1)} \prod_{j=i+1}^{k} q_j^{m_j} \prod_{j=1}^{i-1} (qq_j)^{\varphi_j(\sigma_1)} \prod_{j=i}^{k} q_j^{\varphi_j(\sigma_1)}.$$

Again, summing over all σ with a given supply vector τ, we get

$$P_s(q; q_2, \ldots, q_k) = \sum_{i=1}^{k} \left(\prod_{j=i+1}^{k} q_j^{m_j} \right) \delta_{s-e_i \geq 0} P_{s-e_i}(q; qq_2, \ldots, qq_{i-1}, q_i, \ldots, q_k).$$

As in the previous section, let

$$P_s(k, I) = P_s(0; \underbrace{0, \ldots, 0}_{I-1}, \underbrace{1, \ldots, 1}_{k-I})$$

and let $J_{s,I} \le I$ be the letter such that $m_i = 0$ for $J_{s,I} < i \le I$, and $m_{J_{s,I}} > 0$ (if $m_1 = \ldots = m_I = 0$, we let $J_{s,I} = 0$). Then, for $I = 1, \ldots, k$, substituting $q = q_2 = \ldots = q_I = 0$, $q_{I+1} = \ldots = q_k = 1$ into the recurrence relation above, we get

$$\begin{aligned} P_s(k, I) = &\sum_{i=1}^{I} \delta_{s-e_i \ge 0} P_{s-e_i}(k, I)\ \delta(m_{i+1} = \ldots = m_I = 0) + \\ &+ \sum_{i=I+1}^{k} \delta_{s-e_i \ge 0} P_{s-e_i}(k, i-1). \end{aligned}$$

Hence, if $J_{s,I} = 0$, the first sum on the right-hand side of the equation is 0, since $m_i = 0$, $i = 1, \ldots, I$, implies that $\delta_{s-e_i \ge 0} = 0$ for all $i \le I$. Let us look at the summands of this sum when $J_{s,I} > 0$. If $J_{s,I} < i \le I$, then $m_i = 0$, so $\delta_{s-e_i \ge 0} = 0$. If $i < J_{s,I}$, then $\delta(m_{i+1} = \ldots = m_I = 0) = 0$, since $m_{J_{s,I}} > 0$. Thus, the only possibly nonzero summand occurs when $i = J_{s,I}$, so our recursive relation takes the form

$$P_s(k, I) = \delta_{J_{s,I} \ge 1} P_{s-e_{J_{s,I}}}(k, I) + \sum_{i=I+1}^{k} \delta_{s-e_i \ge 0} P_{s-e_i}(k, i-1). \tag{3.24}$$

Summing (3.24) over all $\tau \in \mathbb{N}^k$ such that $|\tau| = m_1 + \ldots + m_k = n$, we get

$$\begin{aligned} P_n(k,I) &= \sum_{i=1}^{I} \sum_{\{s\,:\,|s|=n, J_{s,I}=i\}} P_{s-e_i}(k,I) + \sum_{i=I}^{k-1} P_{n-1}(k,i) = \\ &= \sum_{i=1}^{I} P_{n-1}(k-I+i,i) + \sum_{i=I}^{k-1} P_{n-1}(k,i) = \\ &= \sum_{i=1}^{I} P_{n-1}(k-i+1, I-i+1) + \sum_{i=I}^{k-1} P_{n-1}(k,i), \end{aligned} \tag{3.25}$$

which is the same as (3.2).

As in the previous subsection, we also conclude that

$$P_n(k,k-1) = \sum_{m=0}^{n} \binom{n}{m} \binom{m+k-2}{k-2}$$

(we actually count the objects in the same set here), and thus we have

Theorem 3.2 *The number of n-letter words over k-letter alphabet which do not contain the 312 pattern is equal to the number n-letter words over k-letter alphabet which do not contain the 123 pattern, i.e. 1 if $k = 1$, and*

$$u_{k-1,n} = 2^{n-2(k-2)} \sum_{m=0}^{k-2} \sum_{j=m}^{k-2} C_j \binom{2(k-2-j)}{k-2-j} \binom{n+2m}{n}$$

if $k \geq 2$.

As a corollary, we have the main theorem of this chapter:

Theorem 3.3 *All single patterns in S_3 form a single Wilf class on words.*

Chapter 4

Pairs of forbidden 3-letter patterns

Just as in the single pattern case, the reversal and complementation maps break the set of pairs of patterns into equivalence classes, of which there are 6 in this case. We will list only representatives for each class: $\{123, 321\}$, $\{312, 321\}$, $\{123, 231\}$, $\{132, 312\}$, $\{132, 231\}$, $\{132, 213\}$.

In this section, we will still use the Noonan-Zeilberger method but without the heavy machinery which we utilized in the previous chapter.

4.0 A note on further notation

From now on in this part of the thesis, the following notation will be local to the set of forbidden patterns under consideration, unless specifically redefined in a section.

NOTATION 4.1 Let $f^T(n,k)$ be the number of n-long k-ary words which avoid a given set of patterns, i.e. let $f^T(n,k) = |[k]^n(T)|$. We will omit the superscript T and write simply $f(n,k)$ when it is clear what set of patterns is meant. Let $F_k(x) = \sum_{n=0}^{\infty} f(n,k)$ and $F(x,y) = \sum_{k=0}^{\infty} F_k(x)$. Also, let

$$f_i(n,k) = |\{\sigma \in [k]^n(T) \ : \ \sigma(i) = k \text{ and } (\forall m < j)(\sigma(m) \neq k)\}|$$

and

$$f_{i,j}(n,k) = |\{\sigma \in [k]^n(T) \ : \ \sigma(1) = j,\ \sigma(i) = k \text{ and } (\forall m < j)(\sigma(m) \neq k)\}|.$$

Additionally, note that the following initial conditions always hold:

$$f(0,k) = 1, \qquad f(1,k) = k, \tag{IC-k}$$

$$f(n,0) = \delta_{n0}, \quad f(n,1) = 1, \tag{IC-n}$$

$$F_0(x) = 1, \qquad F_1(x) = \frac{1}{1-x}. \tag{IC-F}$$

In general, if the shortest pattern in the set T has length m and the least number of distinct letters in a pattern from the set T is ℓ, then

$$f(n,k) = k^n, \quad \text{if } n \le m \text{ or } k \le \ell. \tag{4.1}$$

Also, we can easily see that

$$\begin{aligned} f(n,k) &= f(n,k-1) + \sum_{i=1}^{n} f_i(n,k) \\ &= f(n,k-1) + f_1(n,k) + \sum_{i=2}^{n} \sum_{j=1}^{k-1} f_{i,j}(n,k), \end{aligned} \tag{4.2}$$

and that if no pattern begins with its greatest letter, then

$$f_1(n,k) = f(n-1,k). \tag{4.3}$$

Finally, let $g_i^T(n,k)$ be the number of words which avoid a set T of patterns and contain exactly i distinct letters. We will omit the T from this notation whenever it is clear what set of patterns is meant. Then

$$f(n,k) = \sum_{i=1}^{k} g_i(n,k), \quad n \ge 1. \tag{4.4}$$

4.1 $\{312, 321\}$

Let $\sigma \in [k]^n(312, 321)$ and suppose $k \ge 3$, $n \ge 0$. If σ begins with the letter 1 or 2, then the rest of σ can be any word which avoids 312 and 321, so the number of words in $[k]^n(312, 321)$ which begin with letters 1 or 2 is $2f(n-1, k)$.

If σ begins with a letter greater than 2, then σ cannot contain both 1 and 2, since that would have resulted in a pattern 312 or a pattern 321 with "3" at position 1. This leaves two possibilities.

Suppose σ does not contain 2. Then σ is a word of length n over the $(k-1)$-letter alphabet $\{1, 3, 4, 5, \ldots, k\}$, which does not begin with 1 and avoids patterns 312 and 321. Since the number of such words which have no restriction on the first letter is $f(n, k-1)$, and the number of those words which do begin with 1 is $f(n-1, k-1)$ (see two paragraphs above), then the number of words σ with don't contain 2 is $f(n, k-1) - f(n-1, k-1)$.

Suppose now that σ does contain 2, then it does not contain 1. It is easy to see that the number of words σ which don't contain 1 is also $f(n, k-1) - f(n-1, k-1)$ (by the argument identical to the one above). Of these, the number of those words which don't contain 1 either is $f(n, k-2)$ (since the first letter being greater than 2 is not a restriction here). Therefore, the number of words σ which don't contain 1 but do contain 2 is $f(n, k-1) - f(n-1, k-1) - f(n, k-2)$.

Thus, we have

$$f(n,k) = 2f(n-1,k) + (f(n,k-1) - f(n-1,k-1)) + \\ + (f(n,k-1) - f(n-1,k-1) - f(n,k-2)), \quad (4.5)$$

which implies that

$$f(n,k) - 2f(n-1,k) - f(n,k-1) = \\ = f(n,k-1) - 2f(n-1,k-1) - f(n,k-2) \quad (4.6)$$

for $k \geq 3$, $n \geq 1$. Thus,

$$f(n,k) - 2f(n-1,k) - f(n,k-1) = \\ = f(n,2) - 2f(n-1,2) - f(n,1) = \\ = 2^n - 2 \cdot 2^{n-1} - 1 = -1. \quad (4.7)$$

for $k \geq 2$ and $n \geq 1$. Notice that (4.7) is also true for $k = 1$, $n \geq 1$. Therefore, multiplying both sides of (4.7) by x^n and summing over $n \geq 1$, we get

$$(F_k(x) - 1) - 2xF_k(x) - (F_{k-1}(x) - 1) = -\frac{x}{1-x},$$

which implies that

$$(1-2x)F_k(x) - F_{k-1}(x) = -\frac{x}{1-x}, \quad k \geq 1, \quad (4.8)$$

so, multiplying both sides of (4.8) by y^k and summing over $k \geq 1$, we get

$$(1-2x)(F(x,y)-1) - yF(x,y) = -\frac{x}{1-x}\frac{y}{1-y}$$

which implies that

$$F(x,y) = \frac{1}{2(1-x)}\left(\frac{1}{1-y} + \frac{(1-2x)^2}{1-2x-y}\right). \tag{4.9}$$

Hence,

$$F_k(x) = \frac{1}{2(1-x)}\left(1 + \frac{1}{(1-2x)^{k-1}}\right) \tag{4.10}$$

and

$$\begin{aligned} f(n,k) &= \delta_{k=\mathrm{odd}} + 2^n \sum_{i=0}^{k-2} (-1)^i \binom{n+k-2-i}{n} \\ &= \delta_{k=\mathrm{odd}} + 2^n \sum_{i=0}^{\lfloor \frac{k-2}{2} \rfloor} \binom{n+k-3-2i}{n-1}. \end{aligned} \tag{4.11}$$

Thus, we have the following

Theorem 4.1 *The number of n-letter words over k-letter alphabet which do not contain either* 312 *or* 321 *pattern is given by the formula (4.11) above.*

4.2 $\{123, 321\}$

Here we can use the famous Erdös-Szekeres theorem which states that given any sequence of $rs+1$ numbers one can choose either an increasing subsequence of $r+1$ numbers or a decreasing subsequence of $s+1$ numbers. Hence, any word having at least $(3-1)^2+1=5$ distinct letters must contain an occurrence of either 123 or 321 pattern. Therefore, any $\sigma \in [k]^n(123,321)$ may contain at most 4 distinct letters, so we have

$$f(n,k) = \sum_{m=1}^{4} g_m(n,k), \quad n \geq 1. \tag{4.12}$$

Suppose a word $\sigma \in [k]^n(123,321)$ has exactly 4 distinct letters, $1 \leq j_1 < j_2 < j_3 < j_4 \leq k$. Then no j_2 or j_3 may occur between any j_1 and j_4 , so

$$\sigma \in \{j_2, j_3\}^*\{j_1, j_4\}^*\{j_2, j_3\}^*. \tag{4.13}$$

Notice, however, that since both j_1 and j_4 must occur in the middle part of σ, j_2 and j_3 cannot both occur before or after them. This implies that

$$\sigma = \underbrace{j_2 \ldots j_2}_{n_1}\underbrace{\sigma_1}_{n_2}\underbrace{j_3 \ldots j_3}_{n_3} \quad \text{or} \quad \sigma = \underbrace{j_3 \ldots j_3}_{n_1}\underbrace{\sigma_1}_{n_2}\underbrace{j_2 \ldots j_2}_{n_3} \tag{4.14}$$

for some $n_1, n_2, n_3 \geq 1$ such that $n_1+n_2+n_3 = n$ and $\sigma_1 \in \{j_1, j_4\}^*$ which contains both j_1 and j_4, i.e. is nonconstant (recall that each letter must appear at least once). Since there are $\binom{k}{4}$ ways to pick j_i's from $[k]$, we have that

$$g_4(n,k) = 2\binom{k}{4} \sum_{\substack{n_1,n_2,n_3 \geq 1 \\ n_1+n_2+n_3=n}} (2^{n_2}-2) = 4\binom{k}{4}\left(2^{n-1} - \binom{n}{2} - 1\right) \tag{4.15}$$

for $n \geq 1$ (clearly $g_m(0,k) = 0$ for $m > 0$).

Suppose now that a word $\sigma \in [k]^n(123, 321)$ contains exactly 3 distinct letters, $1 \leq j_1 < j_2 < j_3 \leq k$. Then there may not be a j_2 between any j_1 and any j_3 as that would produce an occurrence of either a 123 or a 321 pattern. Hence,

$$\sigma = \underbrace{j_2 \dots j_2}_{n_1} \underbrace{\sigma_1}_{n_2} \underbrace{j_2 \dots j_2}_{n_3} \tag{4.16}$$

for some $n_1, n_2, n_3 \geq 1$ such that $n_1 + n_2 + n_3 = n$ and a nonconstant $\sigma_1 \in \{j_1, j_3\}^*$. Since there are $\binom{k}{3}$ ways to pick j_i's from $[k]$, we have that

$$g_3(n,k) = \binom{k}{3} \sum_{\substack{n_1,n_2,n_3 \geq 1 \\ n_1+n_2+n_3=n}} (2^{n_2} - 2) = 2\binom{k}{4}\left(2^{n-1} - \binom{n}{2} - 1\right) \tag{4.17}$$

for $n \geq 1$.

If σ contains less than 3 distinct letters, then forbidden 3-letter patterns place no restriction on σ, so

$$g_2(n,k) = \binom{k}{2}(2^n - 2), \qquad g_1(n,k) = k \tag{4.18}$$

Thus, combining (4.12), (4.15), (4.17) and (4.18), we get

$$\begin{aligned} f(n,k) &= \left(4\binom{k}{4} + 2\binom{k}{3}\right)\left(2^{n-1} - \binom{n}{2} - 1\right) + \binom{k}{2}(2^n - 2) + k = \\ &= \left(2\binom{k}{4} + \binom{k+1}{3}\right)(2^n - 2) - 2\left(2\binom{k}{4} + \binom{k}{3}\right)\binom{n}{2} + k \end{aligned} \tag{4.19}$$

for $n \geq 1$ and $f(0,k) = 1$. Thus, we have the following

THEOREM 4.2 *The number of n-letter words over k-letter alphabet which do not contain either* 123 *or* 321 *pattern is given by the formula (4.19) above.*

4.3 $\{213, 231\}$

Again, (4.3) is true for this set of patterns since neither pattern starts with 3. For this section, we will redefine $f_{i,j}(n,k)$ to be the number of those words σ which avoid both 213 and 231 patterns and have leftmost letter k at position $i \geq 2$ and $\sigma(i-1) = j \leq k-1$ (rather then $j = \sigma(1)$ as in the original definition). Since all the $f_{i,j}(n,k)$ again count elements of mutually disjoint sets, we have that (4.2) holds as well.

Fix $2 \leq i \leq n$ and $1 \leq j \leq k-1$, and let σ be a word counted by $f_{i,j}(n,k)$. Then $\sigma = \sigma_1 j k \sigma_2$ for some words $\sigma_1, \sigma_2 \in [k]^n(213, 231)$ of lengths $i-2$ and $n-i$, respectively. Since σ avoids 213, $\sigma_1 j$ should be a nondecreasing sequence of letters (otherwise, we'll get an occurrence of the 213 pattern with "3" at position i), so σ_1 is a nondecreasing subsequence of letters $\leq j$. Since σ avoids 231, σ_2 may contain only letters $\geq j$ (otherwise, $\sigma(m) < j$ for some $m > i$, and we get an occurrence of the 231 pattern at positions $i-1$, i and m).

Conversely, a similar argument shows that if we pick any σ_1 and σ_2 as described above, then $\sigma = \sigma_1 jk\sigma_2$ avoids both 213 and 231. Hence, we have

$$f(n,k) - f(n,k-1) - f(n-1,k) = \\ = \sum_{i=2}^{n}\sum_{j=1}^{k-1}\binom{i-2+j-1}{j-1} f(n-i,k-j+1), \quad (4.20)$$

which, together with the initial conditions (IC-F), implies that, for $k \geq 1$,

$$\begin{aligned}(1-x)F_k(x) - F_{k-1}(x) = x^2 \sum_{j=1}^{k-1} \frac{1}{(1-x)^j} F_{k-j+1}(x) = \\ = \frac{x^2}{1-x}\sum_{j=2}^{k}\frac{1}{(1-x)^{k-j}}F_j(x) = \quad (4.21) \\ = \frac{x^2}{1-x}\left(\sum_{j=0}^{k}\frac{1}{(1-x)^{k-j}}F_j(x) - \frac{2}{(1-x)^k}\right).\end{aligned}$$

Therefore, multiplying both sides of (4.21) by y^k, summing over $k \geq 1$ and taking into account the initial conditions, we obtain that

$$(1-x)(F(x,y)-1) - yF(x,y) = \\ = x^2\left(\frac{1}{1-x-y}F(x,y) - \frac{1}{1-x} - \frac{2}{1-x-y} + \frac{2}{1-x}\right),$$

which yields

$$F(x,y) = \frac{1}{2(1-x)}\left(\frac{1}{1-y} + \frac{(1-2x)^2}{1-2x-y}\right). \quad (4.22)$$

Hence,

$$F_k(x) = \frac{1}{2(1-x)}\left(1 + \frac{1}{(1-2x)^{k-1}}\right) \quad (4.23)$$

and

$$f(n,k) = \delta_{k=\mathrm{odd}} + 2^n \sum_{i=0}^{k-2} (-1)^i \binom{n+k-2-i}{n}$$
$$= \delta_{k=\mathrm{odd}} + 2^n \sum_{i=0}^{\lfloor \frac{k-2}{2} \rfloor} \binom{n+k-3-2i}{n-1}. \tag{4.24}$$

Thus, we have the following

THEOREM 4.3 *The number of n-letter words over k-letter alphabet which do not contain either* 213 *or* 231 *pattern is given by the formula (4.24) above and the sets of forbidden patterns* $\{312, 321\} \sim \{123, 213\}$ *and* $\{213, 231\}$ *are Wilf-equivalent.*

An interested reader is invited to provide a bijective proof of the above theorem.

4.4 $\{132, 231\}$

Let $i > 1$, then $\sigma = \sigma_1\, k\, \sigma_2$ where σ_1 is an $(i-1)$-letter word over $(k-1)$-letter alphabet and σ_2 is an $(n-i)$-letter word over k-letter alphabet. Suppose that σ_1 contains a letter $j = 0, \ldots, k-1$. Then σ_2 may contain only letters j and k, since otherwise σ would contain either a 132 pattern or a 231 pattern. If $\sigma_2 = k \ldots k$, then the only restriction on σ_1 is that it should avoid 132 and 213 patterns. If, on the other hand, σ_2 contains j as well, then σ_1 may contain only the letter j, i.e. $\sigma_1 = j \dots j$. Thus, for $i = 2, \dots, n$,

$$f_i(n,k) = f(i-1,k-1) + (k-1)(2^{n-i}-1). \tag{4.25}$$

Substituting (4.3) and (4.25) into (4.2), we get

$$\begin{aligned} f(n,k) &= f(n-1,k) + \sum_{i=1}^{n} f(i,k-1) + \sum_{i=2}^{n} (k-1)(2^{n-i}-1) = \\ &= f(n-1,k) + \sum_{i=1}^{n} f(i,k-1) + (k-1)(2^{n-1}-n), \end{aligned}$$

which implies that

$$f(n,k) - 2f(n-1,k) + f(n-2,k) = f(n,k-1) + (k-1)(2^{n-2}-1) \tag{4.26}$$

for $n \geq 2$ and $k \geq 1$, and $f(0,k) = 1$, $f(1,k) = k$, $f(n,0) = \delta_{n0}$. Let $F_k(x) = \sum_{n=0}^{\infty} f(n,k)x^n$, then

$$\begin{aligned} &(F_k(x) - kx - 1) - 2x(F_k(x) - 1) + x^2 F_k(x) = \\ &= (F_{k-1}(x) - (k-1)x - 1) + (k-1)x^2 \left(\frac{1}{1-2x} - \frac{1}{1-x} \right), \end{aligned}$$

or,

$$(1-x)^2 F_k(x) = F_{k-1}(x) - x + \frac{(k-1)x^3}{(1-2x)(1-x)} \tag{4.27}$$

for $k \geq 1$, and $F_0(x) = 1$. Let $F(x,y) = \sum_{k=0}^{\infty} F_k(x) y^k$, then, summing (4.27) over k, we get

$$(1-x)^2 (F(x,y) - 1) = yF(x,y) - \frac{xy}{1-y} + \frac{x^3}{(1-2x)(1-x)} \frac{y^2}{(1-y)^2},$$

which implies that

$$\begin{aligned} F(x,y) &= \frac{(1-x)^2}{((1-x)^2-y)} - \frac{xy}{((1-x)^2-y)(1-y)} + \\ &+ \frac{x^3}{(1-2x)(1-x)} \frac{y^2}{((1-x)^2-y)(1-y)^2} \end{aligned} \tag{4.28}$$

Since

$$\frac{y}{((1-x)^2-y)(1-y)} = \frac{1}{1-(1-x)^2}\left(\frac{(1-x)^2}{(1-x)^2-y} - \frac{1}{1-y}\right),$$

it follows, after some technical manipulations, that

$$\begin{aligned} F_k(x) &= \frac{1}{(1-x)^{2k}} + \left(1 - \frac{1}{(1-x)^{2k}}\right)\frac{1}{x(2-x)}\left(1 - \frac{x^2(1-x)}{(1-2x)(2-x)}\right) + \\ &+ \frac{kx^2}{(1-2x)(1-x)(2-x)} \end{aligned} \tag{4.29}$$

We can get rid of $2-x$ in the denominator, then (4.29) takes the form

$$F_k(x) = \frac{1}{1-x)^{2k-1}} - \sum_{j=1}^{k-1} \frac{x}{(1-x)^{2j}} + \frac{1}{1-2x}\sum_{j=1}^{k-1}\frac{(k-j)x^3}{(1-x)^{2j+1}}, \tag{4.30}$$

so

$$\begin{aligned} f(n,k) = \binom{n+2k-2}{n} - \sum_{j=1}^{k-1}\binom{n+2j-2}{n-1} + \\ + \sum_{j=1}^{k-1}\sum_{i=0}^{n}(k-j)\,2^{n-i}\binom{i-3+2j}{2j} \end{aligned} \tag{4.31}$$

Thus, we have

Theorem 4.4 *The number of n-letter words over k-letter alphabet which do not contain either 132 or 231 pattern is given by the formula (4.31) above.*

4.5 $\{132, 213\}$

Here we see that both equations (4.2) and (4.3) hold again since no pattern begins with 3.

Let σ be a word counted by $f_{i,j}(n,k)$. Then $\sigma = j\sigma_1 k\sigma_2$, where σ_1 and σ_2 are words of length $i-2$ and $n-i$, respectively. Since σ avoids 213, σ_1 should be a non-decreasing sequence of letters j through $k-1$. On the other hand, since σ avoids 132 and σ_2 should not contain letters $j+1$ through $k-1$, and thus, is a sequence of letters $1,\dots,j,k$ which avoids 132 and 213 (as a subword of σ).

On the other hand, if σ_1 and σ_2 are any two words subject to the respective restrictions above, and $\sigma = j\sigma_1 k\sigma_2$, then σ avoids 132 pattern and may contain an occurrence of 213 pattern only if it contains a subsequence of the type ak for some $a < j$. Thus, we have that $\sigma_2 = \sigma_3\sigma_4$, where σ_3 is a (possibly empty) word containing only letters j and k and σ_4 is a word containing only letters $1,\dots,j$ which does not begin with j and avoids patterns 132 and 213.

Conversely, if σ_1, σ_3 and σ_4 are as described above, then it is easy to see that $\sigma = j\sigma_1 k\sigma_3\sigma_4$ avoids 132 and 213.

Suppose that σ_1 has length i_1. The number of words σ_1 is $\binom{i-2+k-j-1}{i-2}$, the number of words σ_3 is 2^{i_1}, and the number of words σ_4 is

$$f(n-i-i_1,j) - f_1(n-i-i_1,j) = f(n-i-i_1,j) - f(n-i-i_1-1,j),$$

hence

$$\begin{aligned} f_{i,j}(n,k) = \sum_{i_1=0}^{n-i} \binom{i-2+k-j-1}{i-2} 2^{i_1} \times \\ \times \left(f(n-i-i_1,j) - f(n-n-i_1-1,j)\right), \end{aligned} \tag{4.32}$$

and thus,

$$\begin{aligned} & f(n,k) - f(n-1,k) - f(n,k-1) = \\ &= \sum_{i=2}^{n} \sum_{i_1=0}^{n-i} \sum_{j=1}^{k-1} \binom{i-2+k-j-1}{i-2} 2^{i_1} (f(n-i-i_1,j) - f(n-i-i_1-1,j)) \\ &= \sum_{i=0}^{n-2} \sum_{i_1=0}^{n-2-i} \sum_{j=1}^{k-1} \binom{i+k-j-1}{i} 2^{i_1} (f(n-2-i-i_1,j) - f(n-3-i-i_1,j)), \end{aligned} \tag{4.33}$$

which yields

$$\begin{aligned} (1-x)F_k(x) - F_{k-1}(x) &= (x^2 - x^3) \sum_{j=1}^{k-1} F_j(x) \frac{1}{1-2x} \frac{1}{(1-x)^{k-j}} \\ &= \frac{x^2}{1-2x} \sum_{j=1}^{k-1} F_j(x) \frac{1}{(1-x)^{k-1-j}} \end{aligned} \tag{4.34}$$

Then, multiplying both sides of (4.34) by y^k and summing over $k \geq 1$ (note that (4.34) is true for $k = 1$), we get

$$(1-x)(F(x,y)-1) - yF(x,y) = \frac{x^2}{1-2x} \frac{(1-x)y}{1-x-y} (F(x,y)-1). \tag{4.35}$$

Solving for $F(x,y)$, we finally obtain

$$F(x,y) = \frac{(1-x)^2((1-2x)-(1-x)y)}{(1-x)^2(1-2x)-(1-x)(2(1-2x)+x^2)y+(1-2x)y^2}. \tag{4.36}$$

Thus, we have

Theorem 4.5 *The generating function for the number of n-letter words over k-letter alphabet which avoid both* 132 *and* 231 *patterns is given by the formula (4.36) above.*

4.6 $\{123, 231\}$

Again, both (4.2) and (4.3) hold, so to find a recursive formula in this case we only need to find $f_{i,j}(n,k),\ 2 \le i \le n,\ 1 \le j \le k-1$. Let $\sigma \in [k]^n(123, 231)$ be a word counted by $f_{i,j}(n,k)$ (i.e. σ starts with j and has its leftmost k in position i). Then $\sigma = j\sigma_1 k\sigma_2$, where σ_1 and σ_2 are words of length $i-2$ and $n-i$, respectively.

Since σ avoids pattern 123 and $\sigma(i) = k$, the subword $j\sigma_1$ must be a nonincreasing sequence of letters, so σ_1 must be a nonincreasing sequence of letters $\le j$. Since σ also avoids pattern 231, $\sigma(1) = j$ and $\sigma(i) = k$, the subword σ_2 may contain only letters $\ge j$. In fact, if σ_1 contains letters other that j (i.e. $\sigma_1 \neq jj\dots j$), then σ_2 also must be a nonincreasing sequence of letters for σ to avoid pattern 123. If, on the other hand, $\sigma_1 = jj\dots j$, then σ contains only letters $\geq j$, and all letters $> j$ are in σ_2. Hence, all letters $> j$ must follow in nonincreasing order for σ to avoid 123. But then if we put j anywhere in σ_2, and σ would still avoid both 123 and 231 as there would be no rise before or after any j which does not involve another j. But the number of words in the second case is given by (3.11) with $k-j+1$ substituted for k. Therefore,

$$f_{i,j}(n,k) = \left(\binom{i-2+j-1}{j-1} - 1\right)\binom{n-i+k-j}{k-j} + \\ + \sum_{m=0}^{n-i}\binom{n-i}{m}\binom{m+k-j-1}{k-j-1}, \quad (4.37)$$

so (4.2), (4.3) and (4.37) yield

$$f(n,k) - f(n,k-1) - f(n-1,k) = \\ = (k-1)\binom{n+k-2}{k} - \binom{n+k-1}{k} + n - 1 + \sum_{m=0}^{n-1}\binom{n-1}{m}\binom{m+k-2}{k-2} \quad (4.38)$$

for $n \geq 1$ and $k \geq 1$. This, together with (IC-n), gives

$$(1-x)F_k(x) - F_{k-1}(x) = \\ = \frac{(k-1)x^2}{(1-x)^{k+1}} - \frac{x}{(1-x)^{k+1}} + \frac{x^2}{(1-x)^2} + \frac{x(1-x)^{k-2}}{(1-2x)^{k-1}} \quad (4.39)$$

for $k \geq 1$, which, together with the initial conditions (IC-F), in turn, implies that

$$\begin{aligned}
&(1-x)(F(x,y)-1) - yF(x,y) = \\
&\qquad = \frac{x^2y^2}{(1-x)(1-x-y)^2} - \frac{xy}{(1-x)(1-x-y)} + \\
&\qquad\qquad + \frac{x^2y}{(1-x)^2(1-y)} + \frac{x(1-2x)y}{(1-x)((1-2x)-(1-x)y)},
\end{aligned}$$

so

$$\begin{aligned}
F(x,y) = \frac{1}{(1-x)(1-x-y)} + \frac{x^2y^2}{(1-x)(1-x-y)^3} - \frac{xy}{(1-x)(1-x-y)^2} + \\
- \frac{x}{(1-x)^2(1-y)} + \frac{(1-2x)^2}{x(1-x)((1-2x)-(1-x)y)} - \frac{(1-x)^2}{x(1-x-y)}. \quad (4.40)
\end{aligned}$$

From this, we conclude that

$$\begin{aligned}
F_k(x) = \frac{1}{(1-x)^{k+2}} + \binom{k}{2}\frac{x^2}{(1-x)^{k+2}} - k\frac{x}{(1-x)^{k+2}} - \frac{x}{(1-x)^2} + \\
+ \frac{1}{x}\left(\left(\frac{1-x}{1-2x}\right)^{k-1} - \left(\frac{1}{1-x}\right)^{k-1}\right) \quad (4.41)
\end{aligned}$$

Hence,

$$\begin{aligned}
f(n,k) = \sum_{m=0}^{n-1}\binom{n}{m}\binom{m+k-1}{k-2} + \\
+ \binom{n+k+1}{k+1} + \binom{k}{2}\binom{n+k-1}{k+1} - k\binom{n+k}{k+1} - n. \quad (4.42)
\end{aligned}$$

Thus, we have

Theorem 4.6 *The generating function for the number of n-letter words over k-letter alphabet which avoid both* 123 *and* 231 *patterns is given by the formula (4.42) above.*

Summarizing the developments of this chapter, we have

Theorem 4.7 *There are 5 Wilf classes of forbidden pairs of patterns in S_3, the only nontrivial equivalence being induced by $\{123, 213\} \sim \{213, 231\}$.*

Chapter 5

Triples of forbidden 3-letter patterns

There are 20 possible triples of patterns in S_3, but after factoring by reversal and complementation, we are left with only 6 classes whose representatives are $\{123, 132, 213\}$, $\{123, 132, 231\}$, $\{123, 213, 231\}$, $\{123, 213, 321\}$, $\{132, 213, 321\}$, $\{132, 213, 312\}$. In this section, as before, we will consider the definitions of $f(n, k)$, $f_i(n, k)$, $f_{i,j}(n, k)$, $F_k(x)$ and $F(x, y)$ as local to the set of patterns forbidden.

5.1 $\{123, 132, 213\}$

Since no pattern begins with the letter 3, both (4.2) and (4.3) hold.

Now let us look at $f_{i,j}(n,k)$, $2 \le i \le n$, $1 \le j \le k-1$. Pick a word $\sigma \in [k]^n(123, 132, 213)$ such that $\sigma(1) = j$, $\sigma(i) = k$. Since σ avoids both 123 and 213, we must have that $\sigma(1) = \sigma(m) = j$ for $m < i$. On the other hand, since σ avoids 132, we must have that $\sigma(m) \in [j] \cup \{k\}$ for $m > i$. Also, if $\sigma(m_0) < j$ for some $m_0 > i$, then we must have $\sigma(m) \neq k$ for any $m > m_0$, otherwise σ would contain an occurrence of 213 at positions 1, m_0, m. Thus, we conclude that $\sigma = \sigma_1 k \sigma_2 \sigma_3$, where σ_1 is an $(i-1)$-long constant string of letters j, σ_2 is a (possibly empty) word (say, of length $i_1 \le n-i$) containing only letters j and k, and $\sigma_3 \in [k]^{n-i-i_1}(123, 132, 213)$ does not begin with the letter j (note that this includes the empty word). Conversely, if for any word $\sigma \in [k]^n$ we have that $\sigma = \sigma_1 k \sigma_2 \sigma_3$, where σ_m, $m = 1, 2, 3$, are as defined above, then σ avoids 123, 132 and 213. Since the number of words σ_1 is 1, the number of words σ_2 is 2^{i_1} and the number of words σ_3 is

$$f(n-i-i_1, j) - f_1(n-i-i_1, j) = f(n-i-i_1, j) - f(n-i-i_1-1, j), \quad (5.1)$$

so

$$f_{i,j}(n,k) = \sum_{i_1=0}^{n-i} 2^{i_1}(f(n-i-i_1, j) - f(n-n-i_1-1, j)), \quad (5.2)$$

and thus,

$$
\begin{aligned}
f(n,k) - f(n-1,k) - f(n,k-1) = \\
&= \sum_{i=2}^{n} \sum_{i_1=0}^{n-i} \sum_{j=1}^{k-1} 2^{i_1}(f(n-i-i_1,j) - f(n-i-i_1-1,j)) \\
&= \sum_{i=0}^{n-2} \sum_{i_1=0}^{n-2-i} \sum_{j=1}^{k-1} 2^{i_1}(f(n-2-i-i_1,j) - f(n-i-i_1-3,j)).
\end{aligned}
\tag{5.3}
$$

Since $F_0(x) = 1$ and $F_1(x) = 1/(1-x)$, (5.3) yields

$$
\begin{aligned}
(1-x)F_k(x) - F_{k-1}(x) &= (x^2 - x^3)\sum_{j=1}^{k-1} F_j(x)\frac{1}{1-2x}\frac{1}{1-x} = \\
&= \frac{x^2}{1-2x}\sum_{j=1}^{k-1} F_j(x).
\end{aligned}
\tag{5.4}
$$

Then, multiplying both sides of (5.4) by y^k and summing over $k \geq 1$ (note that (5.3) is true for $k = 1$), we get

$$
(1-x)(F(x,y)-1) - yF(x,y) = \frac{x^2}{1-2x}\frac{y}{1-y}(F(x,y)-1). \tag{5.5}
$$

Solving for $F(x,y)$, we finally obtain

$$
F(x,y) = \frac{(1-x)(1-2x) - ((1-x)(1-2x)+x^2)y}{(1-x)(1-2x) - (1-x)(2-3x)y + (1-2x)y^2}. \tag{5.6}
$$

Thus, we have

Theorem 5.1 *The generating function for the number of n-letter words over k-letter alphabet which simultaneously avoid 123, 132 and 213 patterns is given by the formula (5.6) above.*

5.2 $\{123, 213, 231\}$

Again, both (4.2) and (4.3) hold since no pattern begins with 3. Let σ be a word counted by $f_{i,j}(n,k)$, $2 \le i \le n$, $1 \le j \le k-1$. Since σ avoids both 123 and 213 patterns, and $\sigma(1) = j$, $\sigma(i) = k$, $\sigma(m) \neq k$ for $m < i$, then we must have that $\sigma(m) = j$ for $m < i$. Since σ avoids 231, we must have that $\sigma(m) \ge j$ for $m > i$. On the other hand, using again the fact that σ avoids 123, we find that all letters $> j$ (which may occur only) at positions $> i$ should be in non-increasing order. Thus, we have that $\sigma = \sigma_1 k \sigma_2$, where σ_1 is a constant string of letters j of length $i-1$, and σ_2 is a string of letters j through k of length $n-i$ such that all its letters except j follow in non-increasing order. Conversely, if σ_1 and σ_2 are as described above, then it is easy to check that $\sigma = \sigma_1 k \sigma_2$ avoids 123, 213 and 231. Note that words σ_2 are exactly those counted by $h_1(n,k)$ of equation (3.11) on page 48, so we have that

$$f_{i,j}(n,k) = \sum_{m=0}^{n-i} \binom{n-i}{m} \binom{m+k-j-1}{k-j-1}, \tag{5.7}$$

and hence,

$$
\begin{aligned}
f(n,k) - f(n,k-1) - f(n-1,k) = \\
&= \sum_{i=2}^{n} \sum_{j=1}^{k-1} \sum_{m=0}^{n-i} \binom{n-i}{m} \binom{m+k-j-1}{k-j-1} = \\
&= \sum_{i=2}^{n} \sum_{m=0}^{n-i} \binom{n-i}{m} \binom{m+k-1}{k-2} = \\
&= \sum_{i=0}^{n-2} \sum_{m=0}^{i} \binom{i}{m} \binom{m+k-1}{k-2} = \qquad (5.8) \\
&= \sum_{m=0}^{n-2} \sum_{i=m}^{n-2} \binom{i}{m} \binom{m+k-1}{k-2} = \\
&= \sum_{m=0}^{n-2} \binom{n-1}{m+1} \binom{m+k-1}{k-2} = \\
&= \sum_{m=0}^{n-1} \binom{n-1}{m} \binom{m+k-2}{k-2} - 1.
\end{aligned}
$$

Together with the usual initial conditions

$$
f(0,k) = 1, \quad f(1,k) = k, \quad f(n,0) = \delta_{n0}, \quad f(n,1) = 1 \tag{5.9}
$$

(which make (5.11) true for $n = 1$ as well), this gives us

$$
(1-x)F_k(x) - F_{k-1}(x) = \frac{x(1-x)^{k-2}}{(1-2x)^{k-1}} - \frac{x}{1-x} \tag{5.10}
$$

for $k \geq 2$. Since

$$
F_0(x) = 1, \quad F_1(x) = \frac{1}{1-x}, \tag{5.11}
$$

we note that (5.10) is true for $k = 1$ as well, so summing over $k \geq 0$, we get

$$
(1-x)(F(x,y) - 1) - yF(x,y) = \frac{xy}{1-x} \left(\frac{1-2x}{(1-2x)-(1-x)y} - \frac{1}{1-y} \right),
$$

which implies that

$$F(x,y) = \frac{(1-2x)^2}{x(1-x)((1-2x)-(1-x)y)} - \frac{(1-x)^2}{x(1-x-y)} + \frac{1}{(1-x)(1-y)}. \tag{5.12}$$

Hence,

$$F_k(x) = \frac{1}{x}\left(\left(\frac{1-x}{1-2x}\right)^{k-1} - \left(\frac{1}{1-x}\right)^{k-1}\right) + \frac{1}{1-x}, \tag{5.13}$$

which, in turn, implies that

$$f(n,k) = \sum_{m=0}^{n-1} \binom{n}{m}\binom{m+k-1}{k-2} + 1 \tag{5.14}$$

for $k > 0$ and $f(n,0) = \delta_{n0}$ as we remarked earlier. Thus, we have

THEOREM 5.2 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 123, 213 *and* 231 *patterns is given by the formula (5.14) above.*

5.3 $\{123, 213, 321\}$

Here we will use the same approach as in the section 4.2. Since any sequence of $(3-1)^2+1 = 5$ or more distinct letters has either an increasing or decreasing subsequence of 3 letters, any word σ which avoids 123 and 321 may contain at most 4 distinct letters. Hence, we need only enumerate the subsets of

$[k]^n(123, 213, 321)$ which contain just the words having exactly m distinct letters for $m = 1, 2, 3, 4$. Let $g_m(n, k)$ be the number of such words, then

$$f(n, k) = \sum_{m=1}^{4} g_m(n, k). \tag{5.15}$$

Suppose a word $\sigma \in [k]^n(123, 213, 321)$ has exactly 4 distinct letters, $1 \leq j_1 < j_2 < j_3 < j_4 \leq k$. Consider any 4-letter subsequence of σ in which all 4 distinct letters occur. This is an instance of some pattern in S_4 which itself avoids patterns 123, 213 and 321. But there is only one such pattern, namely 3412. Fix the chosen 4 letters and choose another one (if there are any left). If the letter(s) adjacent to that fifth letter (which must be the same as one of the first four) were (both) different from it, we would obtain an occurrence of a different pattern in S_4 in σ, which is impossible. Thus, we must have

$$\sigma = \underbrace{j_3 \ldots j_3}_{n_1} \underbrace{j_4 \ldots j_4}_{n_2 - n_1} \underbrace{j_1 \ldots j_1}_{n_3 - n_2} \underbrace{j_2 \ldots j_2}_{n - n_3} \tag{5.16}$$

for some $1 \leq n_1 < n_2 < n_3 \leq n - 1$ (recall that each letter must appear at least once). Since there are $\binom{n-1}{3}$ ways to pick n_i's from $[n-1]$ and $\binom{k}{4}$ ways to pick j_i's from $[k]$, we have that

$$g_4(n, k) = \binom{k}{4}\binom{n-1}{3}. \tag{5.17}$$

Suppose now that a word $\sigma \in [k]^n(123, 213, 321)$ contains exactly 3 distinct letters, $1 \leq j_1 < j_2 < j_3 \leq k$. Suppose also that the word σ is a concatenation of r constant subwords (runs) each adjacent pair of which is different (i.e. these runs are maximally constant). Then, clearly, $r \geq 3$, as σ contains 3 distinct letters. Pick two such runs of letters j_1 and j_3, respectively. Then there may not be a run of j_2's between them as that would produce an occurrence either of 123 or of 321 pattern. Hence,

$$\sigma \in \{j_2\}^*\{j_1, j_3\}^*\{j_2\}^*. \tag{5.18}$$

Also, there may not be a run of j_2's to the left of j_1's and j_3's if a run of j_3 follows a run of j_1 as that would produce an occurrence of 213 pattern. This leaves three possibilities:

- there are no j_2's at the beginning of σ, so σ a concatenation of a nonempty string of j_2's and any nonempty string of j_1's and j_3's which contains both of these letters;

- σ starts with a run of j_2's, then either

$$\sigma = \underbrace{j_2 \dots j_2}_{n_1}\underbrace{j_3 \dots j_3}_{n_2-n_1}\underbrace{j_1 \dots j_1}_{n_3-n_2}\underbrace{j_2 \dots j_2}_{n-n_3} \tag{5.19}$$

  for some $1 \leq n_1 < n_2 < n_3 \leq n-1$, or

$$\sigma = \underbrace{j_2 \dots j_2}_{n_1}\underbrace{j_3 \dots j_3}_{n_2-n_1}\underbrace{j_1 \dots j_1}_{n-n_2} \tag{5.20}$$

for some $1 \le n_1 < n_2 \le n-1$.

From that we can easily conclude that

$$\begin{aligned} g_3(n,k) &= \binom{k}{3}\left(\sum_{i=1}^{n-1}(2^i-2) + \binom{n-1}{3} + \binom{n-1}{2}\right) = \\ &= \binom{k}{3}\left(2^n - 2n + \binom{n}{3}\right) \end{aligned} \tag{5.21}$$

If σ contains less than 3 distinct letters, then forbidden 3-letter patterns place no restriction on σ, so

$$g_2(n,k) = \binom{k}{2}(2^n-2), \qquad g_1(n,k) = k \tag{5.22}$$

Combining (5.15), (5.17), (5.21) and (5.22), we get

$$\begin{aligned} f(n,k) &= \binom{k}{4}\binom{n-1}{3} + \binom{k}{3}\left(2^n - 2n + \binom{n}{3}\right) + \binom{k}{2}(2^n-2) + k = \\ &= \binom{k}{4}\binom{n-1}{3} + \binom{k}{3}\binom{n}{3} + \binom{k+1}{3}(2^n-2n) + \binom{k}{2}(2n-2) + k. \end{aligned} \tag{5.23}$$

Thus, we have

Theorem 5.3 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 123, 213 *and* 321 *patterns is given by the formula (5.23) above.*

5.4 $\{123, 132, 231\}$

Again, (4.2) holds, and since no pattern begins with 3, (4.3) holds as well. Let $\sigma \in [k]^n(123, 132, 231)$ be such that it starts with $j \leq k-1$ and its leftmost letter k occurs at position $i \geq 2$. In other words, let σ be counted by $f_{i,j}(n, k)$. Then $\sigma = j\sigma_1 k\sigma_2$ for some σ_1 and σ_2 which avoid 123, 132 and 231.

Since σ avoids 123, $j\sigma_1$ must be a nonincreasing subsequence of letters, so σ_1 must be nonincreasing subsequence of letters $\leq j$. Since σ avoids 132 and 231, σ_2 cannot contain any letters except j and k (since any other letter, together with the first letter j and the leftmost letter k, would produce an occurrence of either 132 or 231 pattern). Furthermore, if σ_1 contains a letter other than j (say, j_1), then σ_2 cannot contain j either, since j_1, the leftmost k and $j \in \sigma_2$ would constitute an occurrence of the 132 pattern.

Hence, either $\sigma_1 = jj \ldots j$, in which case σ_2 may only contain letters j and k, or $\sigma_1 \neq jj \ldots j$ and is nonincreasing, in which case $\sigma_2 = kk \ldots k$. Since σ_1 has length $i-2$ and σ_2 has length $n-i$, we get

$$f_{i,j}(n, k) = \binom{i-2+j-1}{j-1} - 1 + 2^{n-i}, \tag{5.24}$$

so (4.2), (4.3) and (5.24) yield

$$
\begin{aligned}
&f(n,k) - f(n,k-1) - f(n-1,k) = \\
&\qquad = \sum_{i=2}^{n}\sum_{j=1}^{k-1}\left(\binom{i-2+j-1}{j-1} - 1 + 2^{n-i}\right) = \\
&\qquad = \sum_{i=2}^{n}\left(\binom{i-1+k-2}{k-2} + (k-1)(2^{n-i}-1)\right) = \\
&\qquad = \binom{n-1+k-1}{k-1} - 1 + (k-1)(2^{n-1}-n)
\end{aligned}
\tag{5.25}
$$

for $n \geq 1$, $k \geq 1$. Together with the initial conditions (IC-n), multiplying both sides of (5.25) by x^n and summing over $n \geq 1$ yields

$$
\begin{aligned}
&(1-x)F_k(x) - F_{k-1}(x) = \\
&\qquad = \frac{x}{(1-x)^k} - \frac{x}{1-x} + (k-1)\left(\frac{x}{1-2x} - \frac{x}{(1-x)^2}\right) = \\
&\qquad = \frac{x}{(1-x)^k} - \frac{x}{1-x} + (k-1)\frac{x^3}{(1-2x)(1-x)^2}
\end{aligned}
\tag{5.26}
$$

Again, summing over $k \geq 1$ both sides of the (5.26) multiplied by y^k and taking into account the initial conditions (IC-F), we get

$$
\begin{aligned}
&(1-x)(F(x,y)-1) - yF(x,y) = \\
&\qquad = \frac{xy}{1-x-y} - \frac{xy}{(1-x)(1-y)} + \frac{x^3}{(1-2x)(1-x)^2}\frac{y^2}{(1-y)^2},
\end{aligned}
$$

which implies that

$$F(x,y)=\frac{1-x}{1-x-y}+\frac{xy}{(1-x-y)^2}-\frac{1}{1-x-y}+\frac{1}{(1-x)(1-y)}+$$
$$+\frac{x}{(1-2x)(1-x-y)}-\frac{x}{(1-2x)(1-x)(1-y)}-\frac{x^2y}{(1-2x)(1-x)^2(1-y)^2} \tag{5.27}$$

Hence,

$$\begin{aligned}F_k(x)=&\frac{kx}{(1-x)^{k+1}}-\frac{x}{(1-x)^{k+1}}+\frac{1}{1-x}+\\&+\frac{x}{(1-2x)(1-x)^{k+1}}-\frac{x}{(1-2x)(1-x)}-\frac{kx^2}{(1-2x)(1-x)^2},\end{aligned} \tag{5.28}$$

so

$$\begin{aligned}f(n,k)=&(k-1)\binom{n+k-1}{k}+\\&+\sum_{m=0}^{n-1}\binom{m+k}{k}2^{n-1-m}-(k+1)2^n+k(n+1)+2\end{aligned} \tag{5.29}$$

for $k \geq 1$, and $f(n,0)=\delta_{n0}$. Thus, we have

THEOREM 5.4 *The number of n-letter words over k-letter alphabet which simultaneously avoid 123, 132 and 231 patterns is given by the formula (5.29) above.*

5.5 $\{123, 231, 312\}$

Again we write down equation (4.2) and notice that (4.3) no longer holds since 312 starts with 3.

If a word $\sigma \in [k]^n(123, 231, 312)$ starts with the letter k, then its subsequence consisting of letters $\neq k$ must be nonincreasing, or else σ would not avoid the 312 pattern. This leaves two possibilities for σ.

If σ contains at most two distinct letters (i.e. at most one letter distinct from k), then it avoids all 3-letter patterns in S_3. If σ contains at least 3 distinct letters (k and two other ones, say, a and b, $a < b$) then a should follow b, otherwise, kab would be an occurrence of 312 pattern in σ. But then no letter k may occur between any two such letters a and b, otherwise bka would be an occurrence of 231 pattern in σ. Hence, if σ starts with k and contains at least 2 distinct letters other than k, then any letter k should either precede or follow all other letters, which should be arranged in nonincreasing order. Therefore, we have that, for $n \geq 1$ and $k \geq 2$,

$$f_1(n,k) = (k-1)(2^{n-1}-1) + 1 + \sum_{m=0}^{n-1}\sum_{r=1}^{m}\left(\binom{r+k-2}{k-2} - (k-1)\right) =$$

$$= (k-1)(2^{n-1}-1) + 1 + \sum_{m=0}^{n-1}\left(\binom{m+k-1}{k-1} - 1 - (k-1)m\right) =$$

$$= (k-1)\left(2^{n-1} - \binom{n}{2} - 1\right) + \binom{n+k-1}{k} - n + 1 \quad (5.30)$$

Suppose now that σ is counted by $f_{i,j}(n,k)$ for some $i \geq 2$ and $j \leq k-1$. Then σ starts with j and the leftmost letter k in σ occurs in position i, so $\sigma = j\sigma_1 k\sigma_2$ for some words σ_1 and σ_2. Since σ avoids 123 pattern, $j\sigma_1$ must be a nonincreasing sequence of letters, so σ_1 is a nonincreasing sequence of letters $\leq j$. Since σ avoids a 231 pattern, the word σ_2 must contain only letters $\geq j$ (otherwise, $\sigma(m) < j$ for some $m > i$, so σ contains an occurrence of 231 pattern at positions 1, i, m). Since σ_2 avoids 312 pattern and is preceded by k, all letters $\neq k$ of σ_2 must be in nonincreasing order.

Clearly, if σ_1 and σ_2 are nonincreasing subsequences of letters $\leq j$ and $\geq j$, respectively, then an easy case-by-case analysis shows that $\sigma = j\sigma_1 k\sigma_2$ avoids 123, 231 and 312. Since σ_1 should always be a nonincreasing subsequence of letters, consider a case when σ_2 contains an increase, say, $j \leq \sigma(i_1) < \sigma(i_2) \leq k$ for some $i < i_1 < i_2 \leq n$. Then, by the preceding paragraph, we must have $\sigma(i_2) = k$. But $\sigma(1) = j$, so in order for σ to avoid 123 pattern, we must have $\sigma(i_1) = j$. Then, to avoid 123 pattern we must have $\sigma(r) = j$ for all $r < i$ (i.e. $\sigma_1 = jj\ldots j$) and $\sigma(r) \in \{j,k\}$ for all $i \leq r \leq i_2$. Also, we have $\sigma(i) = k$, $\sigma(i_1) = j$, so in order for σ to avoid 312 patterns, no letters except k and j may appear in position $r \geq i_1$.

Hence, either σ_1 and σ_2 are nonincreasing subsequences of letters $\leq j$ and $\geq j$, respectively, or σ contains only letters j and k. These two possibilities are not mutually exclusive, and the words σ which satisfy both of these requirements are

$$\sigma = \underbrace{jj\ldots j}_{i-1}\underbrace{kk\ldots k}_{t}\underbrace{jj\ldots j}_{n-i+1-t}, \quad 1 \leq t \leq n-i+1. \tag{5.31}$$

Thus,

$$f_{i,j}(n,k) = \binom{i-2+j-1}{j-1}\binom{n-i+k-j}{k-j} - \binom{n-i+1}{1} + 2^{n-i} \tag{5.32}$$

Notice that

$$\sum_{i=2}^{n} \binom{i-2+j-1}{j-1}\binom{n-i+k-j}{k-j} = \binom{n-2+k}{k}, \tag{5.33}$$

which follows from the generating function identity

$$x^2 \cdot \frac{1}{(1-x)^j} \cdot \frac{1}{(1-x)^{k-j+1}} = x^2 \cdot \frac{1}{(1-x)^{k+1}}. \tag{5.34}$$

Hence, (4.2), (5.30), (5.32) and (5.33) yield, for $n \geq 1$ and $k \geq 2$,

$$
\begin{aligned}
f(n,k) - f(n,k-1) &= f_1(n,k) + \sum_{j=1}^{k-1}\sum_{i=2}^{n} f_{i,j}(n,k) \\
&= (k-1)(2^{n-1}-1) + 1 + \binom{n+k-1}{k} - (k-1)\binom{n}{2} - n + \\
&\quad + \sum_{j=1}^{k-1}\left(\binom{n-2+k}{k} - \binom{n}{2} + 2^{n-1} - 1\right) \\
&= (k-1)\left(2^n - 2\binom{n}{2} - 2\right) + 1 - n + \binom{n+k-1}{k} + (k-1)\binom{n+k-2}{k} \\
&= (k-1)\left(2^n - 2\binom{n}{2} - 2\right) + 1 - n + \binom{n+k-2}{k-1} + k\binom{n+k-2}{k} \\
&= (k-1)\left(2^n - 2\binom{n}{2} - 2\right) + 1 - n + \binom{n+k-2}{k-1} + (n-1)\binom{n+k-2}{k-1} \\
&= (k-1)(2^n - n^2 + n - 2) + n\binom{n+k-2}{k-1} - n + 1.
\end{aligned}
\tag{5.35}
$$

Hence, summing both sides of (5.35) over the second variable ranging from 2 to k and noting that $f(n,1) = 1$, we get

$$
f(n,k) = \binom{k}{2}(2^n - n^2 + n - 2) + n\binom{n+k-1}{k-1} - k(n-1) \tag{5.36}
$$

for $n \geq 1$, $k \geq 1$, and $f(0,k) = 1$ for $k \geq 0$, $f(n,0) = \delta_{n0}$ for $n \geq 0$. Thus, we have

Theorem 5.5 *The number of n-letter words over k-letter alphabet which simultaneously avoid 123, 312 and 231 patterns is given by the formula (5.36) above.*

5.6 $\{132, 213, 231\}$

As usual, we write down equation (4.2) and notice that (4.3) holds again since none of the three forbidden patterns starts with 3.

Let $\sigma \in [k]^n(132, 213, 231)$ be counted by $f_{i,j}(n, k)$, $i \geq 2$, $j \leq k-1$, i.e. suppose that σ starts with $j < k$ and has its leftmost k in position $i > 1$. We have that $\sigma = j\sigma_1 k\sigma_2$ for some k-ary words σ_1 and σ_2 of lengths $i-2$ and $n-i$, respectively. Since $\sigma(i) = k$ and σ avoids 213 pattern, $j\sigma_1$ must be a nondecreasing sequence of letters $\leq k-1$, so σ_1 must be a nondecreasing sequence of letters from $\{j, j+1, \ldots, k-1\}$. Since $\sigma(1) = j < k$, $\sigma(i) = k$ and σ avoids 132 and 231 patterns, σ_2 may only contain letters j and k. Moreover, if $\sigma_1 \neq jj\ldots j$, then $\sigma_2 = kk\ldots k$, since σ avoids 231 pattern. Therefore,

$$f_{i,j}(n, k) = \binom{i-2+k-j-1}{k-j-1} - 1 + 2^{n-i}. \tag{5.37}$$

But this is almost identical to (5.24), in fact,

$$f_{i,j}^{\{123,132,231\}}(n, k) = f_{i,k-j}^{\{213,132,231\}}(n, k), \tag{5.38}$$

and $1 \leq j \leq k-1$ implies that $1 \leq k-j \leq k-1$ as well. Since both (4.2) and (4.3) apply to both sets of patterns, we conclude that the recursive relations are the same for both sets, which, together with the same initial conditions, gives

$$f^{\{123,132,231\}}(n,k) = f^{\{213,132,231\}}(n,k), \tag{5.39}$$

for all $n \geq 0$, $k \geq 0$. In fact, (5.24), (5.37) and the analyses preceding them suggest a simple bijection between $[k]^n(123,132,231)$ and $[k]^n(213,132,231)$, which acts on each letter of a given word σ of either set as follows: if j is the highest letter of σ which occurs after (but not necessarily immediately after) a smaller letter, then all letters of σ which are $\geq j$ are fixed, and each letter $j_1 < j$ is sent to $j - j_1$. Thus, we have

THEOREM 5.6 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 213, 132 *and* 231 *patterns is given by the formula (5.29), and the sets* $\{123, 132, 231\}$ *and* $\{213, 132, 231\}$ *are Wilf-equivalent.*

Summarizing the developments of this chapter, we have

THEOREM 5.7 *There are 5 Wilf classes of forbidden triples of patterns in S_3, with the only nontrivial equivalence being induced by* $\{123, 132, 231\} \sim \{213, 132, 231\}$.

Chapter 6

Sets of 4 forbidden 3-letter patterns

There are 15 possible quadruples of patterns in S_3, but after factoring by reversal and complementation, we are again left with only 6 classes whose representatives are $\{123, 132, 213, 231\}$, $\{123, 132, 213, 321\}$, $\{123, 132, 231, 312\}$, $\{123, 132, 231, 321\}$, $\{123, 213, 231, 321\}$, $\{132, 213, 231, 312\}$. In this section, as before, we will consider the definitions of $f(n, k)$, $f_i(n, k)$, $f_{i,j}(n, k)$, $g_m(n, k)$, $F_k(x)$ and $F(x, y)$ as local to the set of patterns forbidden.

6.1 $\{123, 132, 213, 231\}$

Since no pattern begins with the letter 3, both (4.2) and (4.3) hold. (Note that this is the only class of quadruples out 6 to be considered in this chapter such that (4.3) holds for one of its respresentatives.)

Let $\sigma \in [k]^n(123, 132, 213, 231)$ be a word which starts with $j \neq k$ and has its leftmost letter k in position $i \geq 2$ (in other words, σ is counted by $f_{i,j}(n,k)$). Then σ may not contain any letters other than j and k since the first letter j, the leftmost letter k and any letter distinct from both of them would constitute an occurrence of a 3-letter pattern in S_3 which does not begin with 3, i.e. 123, 132, 213 or 231. Hence, first i letters of σ are fixed ($\sigma(m) = j$, $m < i$, $\sigma(i) = k$), so we have for $n \geq 2$ and $k \geq 2$.

$$f_{i,j}(n,k) = 2^{n-i}, \tag{6.1}$$

so

$$f(n,k) - f(n,k-1) - f(n-1,k) = \sum_{j=1}^{k-1} \sum_{i=2}^{n} 2^{n-i} = (k-1)(2^{n-1} - 1). \tag{6.2}$$

Note that our initial conditions imply that (6.2) actually holds when $n \geq 1$ and $k \geq 1$. Multiplying both sides by x^n, summing over $n \geq 1$ and substituting (IC-n), we get

$$(1-x)F_k(x) - F_{k-1}(x) =$$
$$= (k-1)\left(\frac{x}{1-2x} - \frac{x}{1-x}\right) = (k-1)\frac{x^2}{(1-2x)(1-x)}. \quad (6.3)$$

Multiplying this through by y^k, summing up over $k \geq 1$ and substituting (IC-F), we get

$$(1-x)(F(x,y)-1) - yF(x,y) = \frac{x^2}{(1-2x)(1-x)}\frac{y^2}{(1-y)^2},$$

which implies that

$$F(x,y) = \frac{2(1-x)^2}{(1-2x)(1-x-y)} - \frac{1}{(1-2x)}\frac{1}{(1-y)^2} - \frac{1}{(1-x)}\frac{y}{(1-y)^2}, \quad (6.4)$$

so

$$F_k(x) = \frac{2}{(1-2x)(1-x)^{k-1}} - \frac{k+1}{(1-2x)} - \frac{k}{(1-x)}, \quad (6.5)$$

and hence,

$$f(n,k) = \sum_{i=0}^{n} 2^{n-i+1}\binom{i+k-2}{i} - (k+1)2^n + k \quad (6.6)$$

for $k \geq 2$, $f(n,0) = \delta_{n0}$, $f(n,1) = 1$. Thus, we have

Theorem 6.1 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 123, 132, 213 *and* 231 *patterns is given by the formula (6.6) above.*

6.2 $\{123, 132, 231, 312\}$

The equation (4.3) does not hold in this case since 312 starts with 3, so we have to find both $f_{i,j}(n,k)$ and $f_1(n,k)$. However, looking at the description of a word $\sigma \in [k]^n(123, 132, 231)$ which starts with $j < k$ and has its leftmost letter k in position i (see section 3.4), we notice that such a word σ should also avoid the 312 pattern. Hence,

$$f_{i,j}^{\{123,132,231,312\}}(n,k) = f_{i,j}^{\{123,132,231\}}(n,k) = \binom{i-2+j-1}{j-1} + 2^{n-i} - 1, \quad (6.7)$$

so (4.2) and (5.25) imply that

$$f(n,k) - f(n,k-1) = f_1(n,k) + \binom{n+k-2}{k-1} - 1 + (k-1)(2^{n-1} - n). \quad (6.8)$$

On the other hand, if a word $\sigma \in [k]^n(123, 231, 312)$ starts with k, then we notice from its description in section 3.5 that should avoid the pattern 132 as well. Hence, we get from (5.30) that

$$f_1^{\{123,132,231,312\}}(n,k) = f_1^{\{123,231,312\}}(n,k) =$$
$$= (k-1)\left(2^{n-1} - \binom{n}{2} - 1\right) + \binom{n+k-1}{k} - n + 1, \quad (6.9)$$

so (6.8) and (6.9) together imply that

$$f(n,k) - f(n,k-1) =$$

$$= (k-1)\left(2^n - \binom{n}{2} - n - 1\right) + \binom{n+k-1}{n-1} + \binom{n+k-2}{n-1} - n \quad (6.10)$$

for $k \geq 1$ and $n \geq 1$. Summing up both sides of (6.10) over the second variable and noting that $f(n,0) = 0$ for $n \geq 1$, we get

$$f(n,k) = \binom{k}{2}\left(2^n - \binom{n}{2} - n - 1\right) + \binom{n+k}{n} + \binom{n+k-1}{n} - kn - 1. \quad (6.11)$$

THEOREM 6.2 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 123, 132, 231 *and* 312 *patterns is given by the formula (6.11) above.*

6.3 $\{123, 132, 213, 321\}$

In this case, both 123 and 321 patterns are forbidden, so any word which avoids them may contain at most 4 distinct letters (see sections 2.2, 3.3). Let us find $g_m(n,k)$, $m = 1, 2, 3, 4$ (see definition of $g_m(n,k)$ on page 67). Let $\sigma \in [k]^n(123, 132, 213, 321)$.

Suppose σ has exactly 4 distinct letters. Since σ avoids 123, 213 and 321, we must have (5.16) from page 87 for some letters $j_1 < j_2 < j_3 < j_4$. But then σ also avoids 132. Conversely, any σ which looks like (5.16) avoids all four patterns in our set. Hence, we have

$$g_4^{\{123,132,213,321\}} = g_4^{\{123,213,321\}} = \binom{k}{4}\binom{n-1}{3}. \tag{6.12}$$

Suppose now that σ contains exactly 3 distinct letters, say, $j_1 < j_2 < j_3$. Then (5.18) from page 88 must hold if σ is to avoid 123 and 321, and σ must either look like (5.19) or (5.20), or be a concatenation of a nonempty and nonconstant string of j_1's and j_3's followed by a nonempty string of j_2's. But in this last case all j_1's must follow all j_3's, so

$$\sigma = \underbrace{j_3 \dots j_3}_{n_1}\underbrace{j_1 \dots j_1}_{n_2-n_1}\underbrace{j_2 \dots j_2}_{n-n_2} \tag{6.13}$$

for some $1 \le n_1 < n_2 \le n-1$. Hence,

$$g_3(n,k) = \binom{k}{3}\left(\binom{n-1}{3} + 2\binom{n-1}{2}\right). \tag{6.14}$$

Since 3-letter patterns don't put restrictions on words which contain less than 3 letters, we have

$$g_2(n,k) = \binom{k}{2}(2^n - 2), \qquad g_1(n,k) = k \tag{6.15}$$

for $n \ge 1$. Thus, (4.12), together with the above, yields

$$f(n,k) = \binom{k+1}{4}\binom{n-1}{3} + 2\binom{k}{3}\binom{n-1}{2} + \binom{k}{2}(2^n-2) + k \tag{6.16}$$

for $n \ge 1$, and $f(0,k) = 1$ for $k \ge 0$.

THEOREM 6.3 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 123, 132, 213 *and* 321 *patterns is given by the formula (6.16) above.*

6.4 $\{123, 132, 231, 321\}$

Again, both 123 and 321 are among the patterns forbidden, so we only need to find $g_m(n, k)$, $m = 1, 2, 3, 4$. In fact, $g_1(n, k) = k$ and $g_2(n, k) = \binom{k}{2}(2^n - 2)$ as in the previous section, since all patterns are in S_3. Thus, we only have to find $g_3(n, k)$ and $g_4(n, k)$. Let $\sigma \in [k]^n(123, 132, 231, 321)$ be a word which contains at least 3 distinct letters.

Suppose that σ contains exactly 4 distinct letters. Pick a subsequence of σ which consists of one copy of each distinct letter of σ. This subsequence is an occurrence of a 4-letter pattern in S_4 which itself must avoid patterns 123, 132, 231 and 321. But there is no such pattern, so σ cannot have 4 distinct letters, and $g_4(n, k) = 0$.

Suppose σ contains exactly 3 distinct letters, say, $j_1 < j_2 < j_3$. Then we again have

$$\sigma \in \{j_2\}^*\{j_1, j_3\}^*\{j_2\}^*. \tag{6.17}$$

If σ starts with a nonempty string j_2's, then all j_3's must follow all j_1's since σ avoids the pattern 231. But then σ cannot have a nonempty string of j_2's at the end since it avoids the pattern 132. On the other hand, if σ does end with a nonempty string of j_2's, then all j_3's must precede all j_1's since σ avoids the pattern 132. Hence, either

$$\sigma = \underbrace{j_2 \ldots j_2}_{n_1} \underbrace{j_1 \ldots j_1}_{n_2-n_1} \underbrace{j_3 \ldots j_3}_{n-n_2} \tag{6.18}$$

or

$$\sigma = \underbrace{j_3 \ldots j_3}_{n_1} \underbrace{j_1 \ldots j_1}_{n_2-n_1} \underbrace{j_2 \ldots j_2}_{n-n_2} \tag{6.19}$$

for some $1 \leq n_1 < n_2 \leq n-1$, so

$$g_3(n,k) = 2\binom{k}{3}\binom{n-1}{2} \tag{6.20}$$

and

$$f(n,k) = 2\binom{k}{3}\binom{n-1}{2} + \binom{k}{2}(2^n-2) + k \tag{6.21}$$

for $n \geq 1$, and $f(0,k) = 1$ for $k \geq 0$.

Theorem 6.4 *The number of n-letter words over k-letter alphabet which simultaneously avoid 123, 132, 231 and 321 patterns is given by the formula (6.21) above.*

6.5 $\{123, 213, 231, 321\}$

The developments in this section repeat those in last one almost literally. Again, we have $g_1(n,k) = k$, $g_2(n,k) = \binom{k}{2}(2^n - 2)$ and $g_4(n,k) = 0$ for the same reasons as in previous section, and if a word $\sigma \in [k]^n(123, 213, 231, 321)$ contains exactly 3 distinct letters, then we again have (6.17) for some $j_1 < j_2 < j_3$. However, in this case, σ may not start with j_2 since it would then contain an occurrence of either 213 or 231 pattern. Hence,

$$\sigma \in \{j_1, j_3\}^*\{j_2\}^* \tag{6.22}$$

and, conversely, any word σ which looks like (6.22) avoids all four patterns in our set. Therefore,

$$g_3(n,k) = \binom{k}{3}\sum_{i=1}^{n-1}(2^i - 2) = \binom{k}{3}(2^n - 2n), \tag{6.23}$$

so

$$\begin{aligned} f(n,k) = \binom{k}{3}(2^n - 2n) + \binom{k}{2}(2^n - 2) + k = \\ = \binom{k+1}{3}(2^n - 2n) + \binom{k}{2}(2n - 2) + k. \end{aligned} \tag{6.24}$$

for $n \geq 1$, and $f(0,k) = 1$ for $k \geq 0$.

THEOREM 6.5 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 123, 213, 231 *and* 321 *patterns is given by the formula (6.24) above.*

6.6 $\{132, 213, 231, 312\}$

Since $g_1(n,k) = k$ and $g_2(n,k) = \binom{k}{2}(2^n - 2)$ for any set of forbidden 3-letter patterns from S_3, we only have to find the number of words which avoid $\{132, 213, 231, 312\}$ and contain at least 3 distinct letters.

Let $\sigma \in [k]^n(132, 213, 231, 312)$ contain at least 3 distinct letters and pick any 3 distinct letters $j_1 < j_2 < j_3$ from σ. The subsequence formed by these letters constitutes an occurrence in σ of some 3-letter pattern in S_3. But then this pattern must be either 123 or 321 since all the other patterns in S_3 are forbidden. Therefore σ must be either a nondecreasing sequence of letters or a nondecreasing one. Hence, the number of such words is

$$g_{\geq 3}(n,k) = 2\left(\binom{n+k-1}{k-1} - \binom{k}{2}(n-1) - k\right), \tag{6.25}$$

the two subtrahends being the numbers of monotone (nondecreasing or nonincreasing) words which contain exactly 2 letters or exactly 1 letter, respectively. Thus,

$$\begin{aligned} f(n,k) &= 2\left(\binom{n+k-1}{k-1} - \binom{k}{2}(n-1) - k\right) + \binom{k}{2}(2^n-2) + k \\ &= 2\binom{n+k-1}{k-1} + \binom{k}{2}(2^n-2n) - k \end{aligned} \tag{6.26}$$

for $n \geq 1$ and $k \geq 1$, and $f(0,k) = 1$ for $k \geq 0$, $f(n,0) = \delta_{n0}$ for $k \geq 0$.

THEOREM 6.6 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 132, 213, 231 *and* 312 *patterns is given by the formula (6.26) above.*

Summarizing the developments of this chapter, we have

THEOREM 6.7 *There are 6 Wilf classes of forbidden quadruples of patterns in S_3.*

Chapter 7

Forbidden sets of 5 and 6 patterns in S_3

7.1 $\{123, 132, 213, 231, 312\}$

Looking at the developments of section 4.6, we can easily see that the words which avoid our set of patterns are all those which avoid $\{132, 213, 231, 312\}$ except the nondecreasing words having 3 or more distinct letters. Thus,

$$\begin{aligned} f(n,k) &= \binom{n+k-1}{k-1} - \binom{k}{2}(n-1) - k + \binom{k}{2}(2^n - 2) + k = \\ &= 2\binom{n+k-1}{k-1} + \binom{k}{2}(2^n - n - 1) \end{aligned} \tag{7.1}$$

for $n \geq 1$, and $f(0,k) = 1$ for $k \geq 0$.

THEOREM 7.1 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 123, 132, 213, 231 *and* 312 *patterns is given by the formula (7.1) above.*

7.2 $\{123, 132, 213, 231, 321\}$

Here the developments repeat those in section 4.5 except that additional restriction of avoiding 132 means that if $\sigma \in \{j_1, j_3\}^*\{j_2\}^*$ for some $1 \le j_1 < j_2 < j_3 \le k$ as in (6.22), then all j_1's must follow all j_3's, so we have (6.19). Thus,

$$f(n,k) = \binom{k}{3}\binom{n-1}{2} + \binom{k}{2}(2^n - 2) + k \tag{7.2}$$

for $n \ge 1$, and $f(0,k) = 1$ for $k \ge 0$.

THEOREM 7.2 *The number of n-letter words over k-letter alphabet which simultaneously avoid* 123, 132, 213, 231 *and* 321 *patterns is given by the formula (7.2) above.*

The next theorem is almost too obvious to state, and we include it only for completeness.

THEOREM 7.3 *There are 2 Wilf classes of forbidden 5-tuples of patterns from S_3.*

7.3 $\{123, 132, 213, 231, 312, 321\} = S_3$

Since all 3-letter patterns in S_3 are forbidden, any word which avoids them may have at most 2 distinct letters, and all words having at most 2 distinct letters avoid all patterns in S_3. Thus,

$$f(n,k) = g_2(n,k) + g_1(n,k) = \binom{k}{2}(2^n - 2) + k \qquad (7.3)$$

for $n \geq 1$, and $f(0,k) = 1$ for $k \geq 0$.

THEOREM 7.4 *The number of n-letter words over k-letter alphabet which simultaneously avoid all patterns in S_3 is given by the formula (7.3) above.*

Chapter 8

Asymptotics

8.1 Common asymptotics

After examining our results for each set of patterns $T \in S_3$, it is not difficult to see that, for $k \geq 2$, each of them is asymptotic to $c(n)2^n$ as $n \to \infty$, where $c(n)$ is some polynomial in n. Hence, we have

THEOREM 8.1 $\lim_{n\to\infty} |[k]^n(T)|^{\frac{1}{n}} = 2$ *for any* $T \subseteq S_3$ *and* $k \geq 2$.

Theorem 8.1 actually follows just from Theorems 3.1, 3.2 and 3.3 of Chapter 3, since any set of patterns is at least as restrictive as any pattern in it.

For general ℓ, we see that the following is almost obvious.

FACT 8.1 $\lim_{n\to\infty} |[k]^n(T)|^{\frac{1}{n}} \geq \ell - 1$ *for any* $T \subseteq S_\ell$ *and* $k \geq \ell - 1$.

This is true because $[k]^n(T)$ always includes all words containing only letters 1 through $\ell - 1$, and there are $(\ell - 1)^n$ such words.

We conjecture that the following stronger statement is true.

CONJECTURE 8.1 $\lim_{n\to\infty} |[k]^n(T)|^{\frac{1}{n}} = \ell - 1$ *for any* $T \subseteq S_\ell$ *and* $k \geq \ell - 1$.

8.2 Asymptotics for the identity pattern

NOTATION 8.1 For two sequences, $\{a_n\}_{n=0}^\infty$ and $\{b_n\}_{n=0}^\infty$, we write $a_n \simeq b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$.

Let us examine asymptotics of the identity pattern in more detail. We recall our main result from Chapter 3.

THEOREM 8.2 *For* $k, n \in \mathbb{N}$, $k \geq 2$, $n \geq 1$, *and every* $\tau \in S_3$,

$$|[k]^n(\tau)| = 2^{n-2(k-2)} \sum_{m=0}^{k-2} \sum_{j=m}^{k-2} c_j d_{k-2-j} \binom{n+2m}{n},$$

where $c_j = \frac{1}{j+1}\binom{2j}{j}$, $d_j = \binom{2j}{j}$.

The highest term in n in the double sum above comes from $c_{k-2}\binom{n+2(k-2)}{n}$ and is equal to

$$c_{k-2}\frac{n^{2(k-2)}}{(2(k-2))!} = \frac{n^{2(k-2)}}{(k-2)!(k-1)!}.$$

Hence, we can write

$$
\begin{aligned}
|[k]^n(\tau)| \simeq 2^{n-2(k-2)} \frac{n^{2(k-2)}}{(k-2)!(k-1)!} &= \\
&= \frac{1!}{(k-2)!(k-1)!} \cdot \left(\frac{1}{2}\right)^{2(k-2)} \cdot n^{2(k-2)} \cdot 2^n \qquad (8.1)
\end{aligned}
$$

as $n \to \infty$ for any $\tau \in S_3$ and any $k \in \mathbb{N}$.

Note that for $\tau = 123 = id_3$ this agrees with the following result of Regev [17]:

Theorem 8.3 (Regev) *For any* $k, \ell \in \mathbb{N}$,

$$
|[k]^n(id_{\ell+1})| \simeq \frac{0!1!2!\dots(\ell-1)!}{(k-\ell)!\dots(k-1)!} \cdot \left(\frac{1}{\ell}\right)^{\ell(k-\ell)} \cdot n^{\ell(k-\ell)} \cdot \ell^n \qquad (8.2)
$$

as $n \to \infty$.

Also, note, after Regev, that

$$
\frac{0!1!2!\dots(\ell-1)!}{(k-\ell)!\dots(k-1)!} = \left(\frac{k!!}{\ell!!(k-\ell)!!}\right)^{-1}, \qquad (8.3)
$$

where we denote $m!! = 0!1!\dots(m-1)!$.

Part III

Single patterns in S_ℓ

Chapter 9

West equivalence

Here we will consider patterns in S_ℓ which are West-equivalent, and prove that West equivalence implies Wilf equivalence for words avoiding permutations, thus extending a theorem of Babson and West [30]. In fact, we will prove a stronger statement, extending this theorem to permutations of multisets avoiding permutations.

A mistake in West's proof of the general case of the sliding lemma [31] discovered late in the writing of this thesis prevents us from proving the general theorem of West [31]. However, we still reduce our corresponding general theorem for words avoiding permutations to West's sliding lemma (which itself appears to be correct, but which this author does not have enough time to prove).

9.1 Words to matrices, patterns to minors

To prove an analog of West's theorem for words avoiding permutations, we need to reformulate our definitions in terms of (0,1)-matrices and their minors.

DEFINITION 9.1 *To each $\sigma \in [k]^n$ we associate a unique $k \times n$, $(0,1)$-matrix M_σ such that $m_{i,j} = \delta_{i,\sigma(j)}$. We will call a (0,1)-matrix which corresponds to a word in such a way, a word matrix. If a word matrix corresponds to a permutation, we will call it a permutation matrix. If a word matrix corresponds to a permutation of a multiset, we will call it a multiset matrix.*

NOTATION 9.1 By abuse of notation, we will denote the set of $k \times n$ word matrices by $[k]^n$, and the set of permutation matrices by S_n.

FACT 9.1 *Any word matrix M_σ contains a single 1 in each column.*

FACT 9.2 *If $\sigma \in [k]^n$ contains all letters of $[k]$, i.e. σ is a permutation of some multiset $\mu = \{1^{\mu_1}, \ldots, k^{\mu_k}\}$ (and we write $\sigma \in S_\mu$), then M_σ (a multiset matrix) also has at least one 1 in each row.*

FACT 9.3 *If $k = n$ and $\sigma \in S_n$, then M_σ has a single 1 in each row and each column.*

DEFINITION 9.2 *A matrix N is said to contain a matrix M if M is a minor of N. Otherwise, N is said to avoid M. We denote the set of word matrices avoiding a matrix M by $[k]^n(M)$, and the set of permutation matrices avoiding a matrix M, by $S_n(M)$.*

FACT 9.4 *M_σ avoids $M_\tau \iff \sigma$ avoids pattern τ.*

9.2 Definition and main theorems

DEFINITION 9.3 *For $1 \leq j \leq \ell$, define $w_j : S_\ell \to S_\ell$ as follows. If $\tau \in S_\ell$ does not begin with $(1, 2, \ldots, j)$ or $(j, j-1, \ldots, 1)$, then $w_j(\tau) = \tau$. Otherwise,*

$$\tau = (1, 2, \ldots, j, a_{j+1}, \ldots, a_k)$$

or

$$\tau = (j, j-1, \ldots, 1, a_{j+1}, \ldots, a_k)$$

for some permutation $(a_{j+1}, \ldots, a_k)$ of $(j+1, \ldots, k)$, and

$$w_j(\tau) = (j, j-1, \ldots, 1, a_{j+1}, \ldots, a_k)$$

or

$$w_j(\tau) = (1, 2, \ldots, j, a_{j+1}, \ldots, a_k),$$

respectively.

DEFINITION 9.4 *We call the equivalence relation induced by* $\{w_1, \ldots, w_\ell\}$ *the West equivalence, and the induced equivalence classes, West classes.*

NOTATION 9.2 We denote the West class of a pattern τ by $\widetilde{\tau}$.

FACT 9.5 *Let* $\tau_1, \tau_2 \in S_\ell$, $\tau_2 = w_t(\tau_1)$ *for some* $t \leq \ell$. *Then*

$$\{M_{\tau_1}, M_{\tau_2}\} = \left\{ \begin{bmatrix} I_t & 0 \\ 0 & A \end{bmatrix}, \begin{bmatrix} J_t & 0 \\ 0 & A \end{bmatrix} \right\},$$

where $A = M_{\tau_1 \ \{t+1,\ldots,\ell\}} = M_{\tau_2 \ \{t+1,\ldots,\ell\}}$.

Here, $\tau \restriction \{t{+}1, \ldots, \ell\}$ denotes the restriction of τ to positions $t{+}1$ through ℓ, I_t is the identity matrix of order t, and J_t is the off-diagonal matrix of order t.

Let us recall the content of West's main theorem in [31] (now a conjecture again for $t \geq 5$):

Conjecture 9.1 (West) *Let M and M' be related by*

$$M = \begin{bmatrix} I_t & 0 \\ 0 & A \end{bmatrix}, \quad M' = \begin{bmatrix} J_t & 0 \\ 0 & A \end{bmatrix},$$

then $|S_n(M)| = |S_n(M')|$ for any $n, t \in \mathbb{N}$. In other words, West-equivalence implies Wilf-equivalence, and each Wilf class contains at least 1 West class.

West proved a special case of this theorem for $t = 2$ earlier in [27], and Babson and West proved this theorem for $t = 3$ in [30]. We extend this theorem to $t = 4$ using the mistake-free part of [31] and make a conjecture for the general case.

Theorem 9.1 *Let M and M' be related by*

$$M = \begin{bmatrix} I_t & 0 \\ 0 & A \end{bmatrix}, \quad M' = \begin{bmatrix} J_t & 0 \\ 0 & A \end{bmatrix},$$

then $|[k]^n(M)| = |[k]^n(M')|$ for any $n \in \mathbb{N}$ and $t \leq 4$.

Conjecture 9.2 *Let M and M' be related by*

$$M = \begin{bmatrix} I_t & 0 \\ 0 & A \end{bmatrix}, \quad M' = \begin{bmatrix} J_t & 0 \\ 0 & A \end{bmatrix},$$

then $|[k]^n(M)| = |[k]^n(M')|$ for any $n, t \in \mathbb{N}$.

In fact, we will prove a stronger statement.

THEOREM 9.2 *Let M and M' be related by*

$$M = \begin{bmatrix} I_t & 0 \\ 0 & A \end{bmatrix}, \quad M' = \begin{bmatrix} J_t & 0 \\ 0 & A \end{bmatrix},$$

then $|S_\mu(M)| = |S_\mu(M')|$ for any multiset $\mu = \{1^{\mu_1}, \ldots, k^{\mu_k}\}$, $\mu_i > 0$, $i = 1, \ldots, k$, when $t \leq 4$.

Again, we conjecture that Theorem 9.2 holds for any $t \in \mathbb{N}$.

To see that Theorem 9.2 implies Theorem 9.1, notice that each supply of letters corresponds to a multiset which, in turn, corresponds to a monotonically nondecreasing pattern on ℓ letters for some $\ell \leq k$, which again corresponds to a multiset on ℓ letters containing all letters 1 through ℓ.

9.3 Reduction to boards

Following West's proof, we will derive theorem 9.2 from a lemma which extends the the class of forbidden objects from multiset matrices to multiset Ferrers boards.

DEFINITION 9.5 *A $(0,1)$-Ferrers board is called a* word board *if it contains a single 1 in each column. A word board is called a* multiset board *if it contains* *at least one 1 in each row. A multiset board is called a permutation board if it contains a single 1 in each row.*

We can define avoidance for boards analogously to avoidance for matrices.

Definition 9.6 *A board ν is said to contain a board μ if μ is a minor of ν. (In other words, we can obtain μ by crossing out some rows and columns of ν.) Otherwise, ν is said to avoid μ.*

Definition 9.7 *A subset $\mathcal{A}$ of a board Q with k rows is said to have content $\gamma(\mathcal{A}) = (\gamma_1(\mathcal{A}), \ldots, \gamma_k(\mathcal{A}))$ if it contains exactly $\gamma_i(\mathcal{A})$ ones in each row $i = 1, \ldots, k$.*

Notation 9.3 We denote the set of multiset boards of shape $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell)$ and content γ avoiding a board ν (i.e. the shape of ν which has the same entires as μ) by $S_{\lambda,\gamma}(\nu)$, and the set of permutation boards of shape λ avoiding a board ν, by $S^0_\lambda(\nu)$.

Here is West's main lemma from [31].

Lemma 9.1 (West) $|S^0_\lambda(I_t)| = |S^0_\lambda(J_t)|$ *for all $t \in \mathbb{N}$ and all shapes λ.*

West's proof turns out to apply only for $t \leq 4$, but the lemma itself is likely to be correct in general case as well.

Our analogous main lemma for multiset boards, to be proved later in the chapter, is as follows.

Lemma 9.2 (Main Lemma) *$|S_{\lambda,\gamma}(I_t)| = |S_{\lambda,\gamma}(J_t)|$ for $t \leq 4$ and all shapes λ and all contents γ.*

Again we conjecture that the lemma is true for any $t \in \mathbb{N}$, which will be proved once its technical sublemma called the sliding lemma is proved for any t.

Our proof that Lemma 9.2 implies Theorem 9.2 closely follows West's proof that Lemma 9.1 implies Conjecture 9.1. In fact, the restriction $t \leq 4$ comes from the sliding lemma and is not used in this proof, which holds for any $t \in \mathbb{N}$. We will try to keep notation the same as often as possible.

Proof [Lemma 9.2 $\implies$ Theorem 9.2].

Let M, M' and μ be as in theorem 9.2 and let $Q = [k] \times [n]$ be the set of squares of the $k \times n$ rectangular board. Pick any $N \in S_\mu(M)$. A square $(i, j) \in Q$ is called *dominant* if N has a minor equal to A in rows $i + 1 \ldots k$ and columns $j + 1 \ldots n$. If (i, j) is dominant, then all (i', j') such that $i' \leq i$ and $j' \leq j$ are also dominant, so the set of dominant squares forms a Ferrers board.

Let $\mathcal{N}(N)$ be the set of *nondominant* squares which contain 1s. Since any dominant square dominates a minimally dominant square (i.e. one which dominates only nondominant squares), it follows that any dominant square dominates a copy of A whose 1s fall entirely in $\mathcal{N}(N)$. Thus, the set of dominant squares of N depends only on $\mathcal{N}(N)$.

For any $\mathcal{A} \subset Q$, let ${}_{\mathcal{A}}S_\mu(M) = \{N \in S_\mu(M) \mid \mathcal{N}(N) = \mathcal{A}\}$. We need to show that $|{}_{\mathcal{A}}S_\mu(M)| = |{}_{\mathcal{A}}S_\mu(M')|$. Then, summing over all subsets $\mathcal{A} \subset Q$, we obtain Theorem 9.2. Since both quantities are zero when $\mathcal{A}$ contains more than one 1 in some column, we may assume that $\mathcal{A}$ contains a single 1 in each column it intersects. (Of course, $\mathcal{A}$ also contains at least one 1 in each row it intersects.)

Suppose $\mathcal{A}$ is as described above, and $|\mathcal{A}| = n - m$. Then, to construct an element of ${}_{\mathcal{A}}S_\mu(M)$, we have to insert the remaining m 1s into the m unclaimed columns and all the unclaimed rows. Moreover, each of these remaining 1s must be inserted into a dominant square by definition of ${}_{\mathcal{A}}S_\mu(M)$. Let $\lambda_{\mathcal{A}}$ be the restriction of of the set of dominant squares onto the unclaimed rows and columns. Note that since the set of all dominant squares forms a Ferrers board, its restriction $\lambda_{\mathcal{A}}$ to the unclaimed rows and columns also forms a Ferrers board and its shape depends only on $\mathcal{A}$.

We obtain an element of ${}_{\mathcal{A}}S_\mu(M)$ by inserting m 1s into the board $\lambda_{\mathcal{A}}$ so that each column contains a single 1 and each row contains at least one 1. (Of course, if this is impossible, i.e. if $\lambda_{\mathcal{A}}$ has $> m$ rows, then $|{}_{\mathcal{A}}S_\mu(M)| = |{}_{\mathcal{A}}S_\mu(M')| = 0$, so suppose $\lambda_{\mathcal{A}}$ has $\leq m$ rows.) Furthermore, we must insert these m 1s so that $\lambda_{\mathcal{A}}$ does not have I_t as a minor. If an I_t were formed, then, together with the copy of A contained in the nondominant squares, it would form a minor isomorphic to M in the resulting $k \times n$ matrix. Conversely, the resulting $k \times n$ matrix may have M as a minor only if there is a copy of I_t contained entirely in the dominant squares. Also, note that $\gamma(\lambda_{\mathcal{A}}) = \mu - \gamma(\mathcal{A})$. Thus, $|{}_{\mathcal{A}}S_\mu(M)| = |S_{\lambda_{\mathcal{A}},\gamma(\lambda_{\mathcal{A}})}(I_t)|$. Similarly, $|{}_{\mathcal{A}}S_\mu(M')| = |S_{\lambda_{\mathcal{A}},\gamma(\lambda_{\mathcal{A}})}(J_t)|$, so Lemma 9.2 implies $|{}_{\mathcal{A}}S_\mu(M)| = |{}_{\mathcal{A}}S_\mu(M')|$, and hence, Theorem 9.2. $\square$

9.4 Proof of main lemma

We begin by restating our main lemma.

Lemma 9.3 (Main Lemma) $|S_{\lambda,\gamma}(I_t)| = |S_{\lambda,\gamma}(J_t)|$ *for all* $t \leq 4$ *and all shapes* λ *and all contents* γ*.*

Even though we stated this lemma only for $t \leq 4$, we will prove that it follows from the sliding lemma for any $t \in \mathbb{N}$.

Again, the proof of this lemma follows West's proof of Lemma 9.1. We partition both $S_{\lambda,\gamma}(I_t)$ and $S_{\lambda,\gamma}(J_t)$ into smaller sets by introducing a list of auxiliary parameters for each case. Then we show that the cardinalities of the corresponding smaller sets are the same for small values of our variables (main and auxiliary) and satisfy the same recurrence relation due to the same transfer of parameters in the induction steps in both cases.

First of all, note that all boards ν in either $S_{\lambda,\gamma}(I_t)$ or $S_{\lambda,\gamma}(J_t)$ are multiset boards, and hence, for either set to be nonempty, λ must have k rows and $\geq k$ columns and dominate a k-staircase, $(k, k-1, \ldots, 2, 1)$ (i.e. $\lambda_j \geq k - j$, $j = 1, \ldots, k$).

We will now define the auxiliary parameters we will need for our proof. They are the same auxiliary parameters as in West's proof, but in our case we need two arrays of these parameters for each of our two cases, I_t-free and J_t-free.

Definition 9.8 *We call a column in a k-row board a full column if it intersects all k rows.*

Definition 9.9 *For any $\nu \in S_{\lambda,\gamma}(I_t)$ [resp. $\nu \in S_{\lambda,\gamma}(J_t)$], let $c^I_x(\nu)$ [resp. $c^J_x(\nu)$], $x = 1, 2, \ldots, t-1, t$, be the largest value of $c \leq \lambda_k$ such that there is* *no I_x [resp. J_x] in the leftmost [resp. rightmost] full columns of ν. Also, let $\mathbf{c}(\nu) = (c_2(\nu), \ldots, c_{t-1}(\nu))$ for both cases.*

Remark 9.1 Clearly, $c_1 = 0$ and $1 \le c_2 \le c_3 \le \ldots \le c_{t-1} \le c_t = \lambda_k$ for any λ and γ in both I-case and J-case.

We will drop the superscripts I and J when it is clear what case is meant or that both cases are meant. We will also drop the ν if we know that it is fixed or that our statement applies to any ν.

Fact 9.6 (West) *If $c_x < \lambda_k$, then $c_x < c_{x+1}$ (for both I-case and J-case). Hence, for any x, $c_x = c_{x+1}$ implies $c_x = c_{x+1} = ... = c_{t-1} = c_t = \lambda_k$.*

Indeed, 1 in column $c_x + 1$ completes an I_x by maximality of c_x, but cannot complete an I_{x+1} since that would imply existence of I_x in columns 1 through c_x. In the J-case, the proof is identical, except we consider columns λ_k through $\lambda_k - c_x$ from the right.

Notation 9.4 Given a board ν, denote by $\bar{\nu}^I$ [resp. $\bar{\nu}^J$] the board obtained by deleting the last row ν_k as well as the column of ν which contained the rightmost [resp. leftmost] 1 in ν_k. Also, let $\bar{\lambda}$ and $\bar{\gamma}$ be the shape and content, respectively, of $\bar{\nu}$.

Note that, if there was more than one 1 in the last row of ν, we do not delete the last row. Again, we will drop the superscript when it is clear which case is meant or that both cases are meant.

NOTATION 9.5 Let

$$g^I_{\lambda,\gamma}(\mathbf{c},\bar{\mathbf{c}}) = \{\nu \in S_{\lambda,\gamma}(I_t) \,:\, \mathbf{c}^I(\nu) = \mathbf{c},\, \mathbf{c}^I(\bar{\nu}^I) = \bar{\mathbf{c}}\},$$

$$g^J_{\lambda,\gamma}(\mathbf{c},\bar{\mathbf{c}}) = \{\nu \in S_{\lambda,\gamma}(J_t) \,:\, \mathbf{c}^J(\nu) = \mathbf{c},\, \mathbf{c}^J(\bar{\nu}^J) = \bar{\mathbf{c}}\}.$$

REMARK 9.2 Note that

$$|S_{\lambda,\gamma}(I_t)| = \sum_{\mathbf{c},\bar{\mathbf{c}}} |g^I_{\lambda,\gamma,}(\mathbf{c},\bar{\mathbf{c}})|,$$

$$|S_{\lambda,\gamma}(J_t)| = \sum_{\mathbf{c},\bar{\mathbf{c}}} |g^J_{\lambda,\gamma}(\mathbf{c},\bar{\mathbf{c}})|.$$

By the above remark, the following lemma implies our main lemma 9.3.

LEMMA 9.4 *For any* λ,γ *and any* $\mathbf{c},\bar{\mathbf{c}}$, *we have* $|g^I_{\lambda,\gamma}(\mathbf{c},\bar{\mathbf{c}})| = |g^J_{\lambda,\gamma}(\mathbf{c},\bar{\mathbf{c}})|$.

Proof. As we stated before, we will prove this lemma by induction. The induction base follows easily from the fact that there is a unique empty board. Now consider the induction step. To prove equality asserted in the lemma, we only need to prove that the recursion relations for each of the two cases is the same. *Note that we don't have to find the actual form of the recursion relation in either case, just show that they are the same.*

In the I-case, we start with $\bar{\nu} \in S_{\bar{\lambda},\bar{\gamma}}(I_t)$, and insert either

(I-a) a new full column with 1 in its last row such it is to the right of all the 1s in the last row, or

(I-b) a new last row and a new full column with 1 in their intersection.

In the J-case, we start with $\bar{\nu} \in S_{\bar{\lambda},\bar{\gamma}}(J_t)$, and insert either

(J-a) a new full column with 1 in its last row such it is to the left of all the 1s in the last row, or

(J-b) a new last row and a new full column with 1 in their intersection.

Then we will show that each of these operations [(a) or (b)] changes $(\mathbf{c}, \bar{\mathbf{c}})$ in the same way in both cases [I or J], i.e. that the transfer of parameters is the same. (We will need a technical sublemma in case (b).) Then we can use this transfer of parameters to calculate the contribution of each $g_{\bar{\lambda},\bar{\gamma}}$ to each $g_{\lambda,\gamma}$ and invert the resulting set of equations to get a formula for each $g_{\lambda,\gamma}$ in terms of $g_{\bar{\lambda},\bar{\gamma}}$. But these formulas should be the same in the I-case and the J-case because of the same transfer of parameters, which completes our induction step.

Case (a). Consider a board $\nu^I \in g^I_{\lambda,\gamma}(\mathbf{c}, \bar{\mathbf{c}})$ and a board $\nu^J \in g^J_{\lambda,\gamma}(\mathbf{c}, \bar{\mathbf{c}})$. We will now insert a new full column with 1 in the last row into ν^I and ν^J.

It is clear from the definitions of $\mathbf{c}$ and $\bar{\mathbf{c}}$ for both cases that inserting this column as the j-th full column (from the left) in ν^I changes $\mathbf{c}^I(\nu^I)$ and $\bar{\mathbf{c}}^I(\nu^I)$ in the same way as inserting it as the $(\lambda_k + 1 - j)$-th full column (j-th full column from the right) ν^J changes $\mathbf{c}^J(\nu^J)$ and $\bar{\mathbf{c}}^J(\nu^J)$. Hence, the transfer of parameters is the same in both I-case and J-case. This proves the induction step for the case (a).

Case (b). This case is a little more subtle because we also insert the new last row of length λ_{k+1} which may be shorter than λ_k, so not all the old full columns become the new full columns. Hence, the proof of case (a) is insufficient for this situation. To use it, we need a sublemma for the case (J-b), called the *sliding lemma*. It states that the simultaneous I_x-freedom in columns $\lambda_{k+1} - c_x + 1$ through λ_{k+1} for $x = 1, \ldots, t$, is equally restrictive as the simultaneous I_x-freedom in columns $\lambda_k - c_x + 1$ through λ_k for $x = 1, \ldots, t$ (when $c_x \leq \lambda_{k+1}$ for $x = 1, \ldots, t-1$). Assuming the sliding lemma (which we will prove next), we can again use the same argument as in case (a) to complete our induction step. $\square$

This concludes the proof of the main lemma.

REMARK 9.3 Note that in case (b), we have $\bar{\mathbf{c}}_{new} = \mathbf{c}_{old}$, and $\mathbf{c}_{new}$ depends only on $\mathbf{c}_{old}$. Thus, we need $\bar{\mathbf{c}}_{old}$ only in case (a), where it is not difficult to see that we do need it, but *not* in case (b), where $\bar{\mathbf{c}}_{old}$ is discarded.

What this means is that in case (b), given fixed λ and γ, the contribution to $g_{\lambda,\gamma}(\mathbf{c},\bar{\mathbf{c}})$ of each nonzero $g_{\bar\lambda,\bar\gamma}(\bar{\mathbf{c}},\bar{\bar{\mathbf{c}}})$ is the same whatever $\bar{\bar{\mathbf{c}}}$ is.

NOTATION 9.6 Let $\tilde{g}_{\lambda,\gamma}(\mathbf{c}) = \bigcup_{\bar{\mathbf{c}}} g_{\lambda,\gamma}(\mathbf{c},\bar{\mathbf{c}})$, i.e.

$$\tilde{g}^I_{\lambda,\gamma}(\mathbf{c}) = \{\nu \in S_{\lambda,\gamma}(I_t) \,:\, \mathbf{c}^I(\nu) = \mathbf{c}\},$$

$$\tilde{g}^J_{\lambda,\gamma}(\mathbf{c}) = \{\nu \in S_{\lambda,\gamma}(J_t) \,:\, \mathbf{c}^J(\nu) = \mathbf{c}\}.$$

By Remark 9.3, in case (b), we only have to deal with contributions of $\tilde{g}_{\bar\lambda,\bar\gamma}(\bar{\mathbf{c}})$ to $g_{\lambda,\gamma}(\mathbf{c},\bar{\mathbf{c}})$, so in this case we only need to prove that the transfer of parameters for $\tilde{g}_{\lambda,\gamma}(\mathbf{c})$ is the same in both I- and J-case at each step (b) we make. Thus, the sliding lemma we need is essentially the same as in [31].

9.5 Sliding lemma

The statement and proof of the sliding lemma for $t \leq 4$ almost literally repeats those in West [31]. As we describe it, we will also mention where his proof fails in general.

We will need the following additional notation and definitions.

NOTATION 9.7 For any multiset board of shape λ and content γ and arrays $\mathbf{c}$ as in Remark 9.1 and Fact 9.6, let

$$h^J_{\lambda,\gamma}(\Delta;\mathbf{c}) = \{\nu \in S_{\lambda,\gamma}(J_t) : \nu \text{ is } J_x\text{-free in columns} \\ \lambda_k - \Delta - c_x + 1 \text{ through } \lambda_k - \Delta\}$$

and $h^J_{\lambda,\gamma}(\mathbf{c}) = h^J_{\lambda,\gamma}(0;\mathbf{c})$.

LEMMA 9.5 (SLIDING LEMMA) *$h^J_{\lambda,\gamma}(\Delta;\mathbf{c})$ is in bijection with $h^J_{\lambda,\gamma}(\mathbf{c})$ for any $\Delta \leq \lambda_k$.*

Notice that in the definition of $g^J_{\lambda,\gamma}(\mathbf{c},\bar{\mathbf{c}})$ we required that boards in it be *maximally* J_x-free on corresponding *strips* (sets of adjacent columns). However, if we define $\tilde{g}^J_{\lambda,\gamma}(\Delta;\mathbf{c})$ similarly with the added maximality condition, it is clear that we obtain $\tilde{g}$ and $\tilde{g}(\Delta;\cdot)$ from h and $h(\Delta;\cdot)$, respectively, through parallel inclusion-exclusion arguments.

Clearly, Lemma 9.5 follows from repeated applications of the following lemma.

LEMMA 9.6 *$h^J_{\lambda,\gamma}(\Delta;\mathbf{c})$ is in bijection with $h^J_{\lambda,\gamma}(\Delta-1;\mathbf{c})$ for any $1 \leq \Delta \leq \lambda_k$.*

In other words, given a nested set of (up to) $t-2$ strips with the common right edge such that the xth strip from the right is J_{x+1}-free, we can slide it one full column to the right. An interested reader may first want to look at the treatment of a special case when there is only 1 strip, i.e. when $t = 3$. It is given (in case of permutations avoiding patterns) in [31] and [30].

DEFINITION 9.10 *We will refer to the* $c_{t-1}+1$ *columns containing our nested set of columns plus one column to their right as the active strip. We will also refer to the* 1 *initially in the rightmost column of the active strip as the extra* 1.

Proof. (of Lemma 9.6) To prove the lemma, we will show that we can pass the extra 1 bijectively through the active strip in several stages until it emerges on the left side of our nested set of strips, while sliding the boundaries of all our strips one column to the right, thus proving the lemma.

West's passing procedure is defined inductively as follows.

We partition our active strip into $\leq t-2$ contiguous non-overlapping strips, which we call the *induced* strips, the first of which is the J_2-free strip, and for $x > 1$, the xth induced strip is the complement of the J_x-free strip in the J_{x+1}-free strip. Note that all of these strips must be nonempty.

Suppose we passed our extra 1 through $x-1$ induced strips, and it is now in the column immediately to the right of the xth induced strip (and immediately to the left of the $(x-1)$-st induced strip, when $x > 1$). There are two possibilities at this point.

A. *The extra* 1 *is not above any* 1 *in the* x*th induced strip.* In this case, we slide the boundaries of all induced strips, starting with xth, one position to the right and declare the 1 in the column immediately to the left of the whole active strip to be the (new) extra 1. This ends the passing procedure.

B. *There are* 1*s in the* x*th induced strip which are strictly lower than the extra* 1. In this case, the column with the extra 1 is removed and inserted in place of the leftmost column in the xth strip which contains a 1 (strictly) lower than the extra 1, and that displaced column is inserted immediately to the left of the xth induced strip (and immediately to the right of the $(x+1)$-st induced strip if such exists) and the 1 in this column is declared to be the (new) extra 1. At this point, if the (new) extra 1 is not to the left of the whole active strip, we repeat the passing procedure, otherwise, we are finished.

Clearly, this procedure is bijective. The inverse procedure is as follows. Given an assemblage of induced strips with an extra 1 on the far left, we first locate the rightmost induced strip (say xth) such that the 1 in its rightmost column is lower than all the other 1s in this induced strip. If there is one, then we slide the boundaries of all induced strips, starting from the xth, one position to the left, otherwise, we do nothing. At this point, if there are strips to the right of the (new) extra 1, we repeat the procedure B backwards until all the induced strips are to the left of the extra 1. In other words, now the column displaced by the extra 1 column is that which contains the rightmost 1 in the xth strip which is (strictly) above the extra 1. This column is moved immediately to the right of xth strip, and the 1 in it is declared the (new) extra 1.

Now we have to prove that, after passing the extra 1 through the whole active strip, the new xth nested strip, i.e. the union of the (new) induced strips 1 through x is still J_{x+1}-free for all $x = 1, \ldots, t_0$, where $t_0 \leq t - 2$ is the number of strips in our active strip. Again, we prove this by induction.

Suppose that we have the extra 1 between $(x-1)$-st and xth induced strips and, for all $y < x$, the new yth nested strip is J_{y+1}-free. (Note that this is vacuously true for $x = 1$.)

Clearly, in our passing procedure we first apply procedure B repeatedly until the extra 1 bottoms out, and then apply procedure A once. As we apply procedure B, the extra 1 is moving downwards through the active strip displacing lower and lower 1s from each successive strip. Until the extra 1 bottoms out, these displaced 1s give an occurrence of J_x for each new xth nested strip and an occurrence of J_{x+1} in the union of xth nested and the extra 1. Also, any resulting occurrences of J_{x+1} in the union of the extra 1 column and the new xth nested strip must contain the extra 1. Hence, all the displaced 1s plus the original extra 1 give us the *worst* contribution that can be made by the union of the xth nested strip and the adjacent extra 1 column on its left. Thus, we can make the induction step in case B.

Now suppose that we have the situation A. This means that the extra 1 is not above any 1 in the x-th induced strip. Simply sliding the boundaries of the all remaining induced strips means leaving our board fixed, so no new J_t's can appear. It is not difficult to see that all c_x-conditions are preserved if the extra 1 bottoms out before the first strip (which is J_2-free), or before the last induced strip (where we only have to slide one induced strip), or after all the strips (in which case the situation A does not even arise).

If the extra 1 bottoms out before the last induced strip then the fact that all c_x-conditions are preserved follows from the argument in case B.

Suppose now that the extra 1 is below all the 1s in the first (J_2-free) strip.

Not that is it means that the union of the first stip and the extra column is also J_2-free and that all the 1s in the first strip are higher up and to the right of the first one. Any possible occurrence of J_{x+1} in some x-th new nested strip must involve the extra 1, so it cannot involve any 1s from the first strip. But then we can replace our extra 1 with any 1 from the first strip to get an occurrence of J_{x+1} in the old x-th nested strip, which contradicts our assumptions.

Clearly, if $t \leq 4$, i.e. when we have ≤ 2 strips, one of these 3 situations obtains, which finishes the proof for $t \leq 4$. $\square$

The problem with this proof arises when we have to slide more than one induced strip after the extra 1 bottoms out. Then, for $y > x$ the leftmost one of the old y-th induced strip becomes the rightmost 1 of the new $(y+1)$-st induced strip. The we may create a J_{y+1} in y-th strip when the original extra 1 was above all the 1s in the first, J_2-free, strip. For example,

$$
\left[\begin{array}{cc|cc|cc|c}
 & & & & & & 1 \\
 & & & 1 & & & \\
 & & 1 & & & & \\
 & & & & 1 & & \\
 & & & & & 1 & \\
1 & & & & & & \\
 & 1 & & & & &
\end{array}\right]
\overset{\text{Step B}}{\rightsquigarrow}
\left[\begin{array}{cc|cc|c|cc}
 & & & & & 1 & \\
 & & & 1 & & & \\
 & & 1 & & & & \\
 & & & & 1 & & \\
 & & & & & & 1 \\
1 & & & & & & \\
 & 1 & & & & &
\end{array}\right]
\overset{\text{Step A}}{\rightsquigarrow}
$$

$$
\overset{\text{Step A}}{\rightsquigarrow}
\left[\begin{array}{c|cc|cc|cc}
 & & & & & 1 & \\
 & & & 1 & & & \\
 & & 1 & & & & \\
 & & & & 1 & & \\
 & & & & & & 1 \\
1 & & & & & & \\
 & 1 & & & & &
\end{array}\right]
\tag{9.1}
$$

9.6 Some corollaries and conjectures

Our versions of West's theorems (Theorems 9.1 and 9.2) for $t \leq 4$ allow us to to find the number of Wilf classes in S_ℓ for $\ell \leq 5$ (for *words* avoiding permutation patterns). Since the cases $\ell = 1, 2$ are trivial and the case $\ell = 3$

τ	1324	1234	1342	1423	2413
$\lvert[7]^7(\tau)\rvert$	647826	647790	647034	647006	646404

Table 9.1: Cardinalities of $[k]^n(\tau)$ for $k = n = 7$, $\tau \in S_4$

.

was covered in section 1.1 of this thesis, only $\ell = 4$ and $\ell = 5$ remain.

The generalization of West's theorem for words (Theorem 9.1) implies that that there are ≤ 5 Wilf classes in S_4. However, the numerical data for 7-letter words over a 7-letter alphabet (see Table 9.1) imply the following corollary.

Corollary 9.1 *There are exactly 5 Wilf classes in S_4 (for words avoiding permutations): $\boxed{\widetilde{1324}}$, $\boxed{\widetilde{1234}}$, $\boxed{\widetilde{1342}}$, $\boxed{\widetilde{1423}}$ and $\boxed{\widetilde{2413}}$. Each Wilf class contains exactly 1 West class.*

Remark 9.4 Notice that each Wilf class in S_4 (for words avoiding permutations) contains exactly one West class, which is not true for permutations. In fact, $\widetilde{1342}^p, \widetilde{2413}^p$ belong to the same Wilf class, and the number of Wilf classes in that case is 3 (see [23], [22] and [27]).

Remark 9.5 Notice also that 1324 is the most avoided pattern in Table 9.6

and it is also the only pattern (up to symmetry) avoided by more words than the identity pattern 1234.

In [1], it is shown that, for all $n \geq 7$, we have $|S_n(1324)| > |S_n(1234)| > |S_n(1342)| = |S_n(1423)| = |S_n(2413)|$. It is not difficult to see that only slight modifications of the proofs of those facts (with essentially the same arguments) will allow us to prove that $|[k]^n(1324)| > |[k]^n(1234)|$ for $n \geq 7$, $|[k]^n(1234)| > |[k]^n(1423)|$ and $|[k]^n(1234)| > |[k]^n(1342)|$ for $n \geq 7$, and $|[k]^n(1234)| = |[k]^n(2143)| > |[k]^n(2413)|$ for $n \geq 7$. Hence, 1324 must be the most avoided pattern by words as well as by permutations.

Similarly, for $\ell = 5$, Theorem 9.1 implies that there are ≤ 24 Wilf classes for words avoiding permutations. On the other hand, our numerical results (see Table 9.2 on page 156) for $n = k = 8$ imply that there are ≥ 21 Wilf classes in S_5. For each West class we list the lexicographically smallest representative.

Remark 9.6 Notice that, for $n = k = 8$, the only 2 cardinality classes in S_5 avoided by more patterns than $\boxed{12345}$ are $\boxed{12435}$ and $\boxed{14325}$.

Among the 21 cardinality classes we get for $n = k = 8$, only 3 may contain ≥ 2 West classes. These are $\boxed{\widetilde{14352}, \widetilde{15243}}$, $\boxed{\widetilde{13542}, \widetilde{15324}}$ and $\boxed{\widetilde{12435}, \widetilde{14325}}$.

However, from the table in [27], p. 47, as well as from [32] (line 3), we have that

$$\begin{aligned}
|S_9(14352)| - |S_9(15243)| &= 260204 - 260202 = 2,\\
|S_9(15324)| - |S_9(13542)| &= 260204 - 260202 = 2, \qquad (9.2)\\
|S_9(14325)| - |S_9(12435)| &= 261863 - 261853 = 10.
\end{aligned}$$

Let us prove that this implies

$$\begin{aligned}
|[9]^9(14352)| &\neq |[9]^9(15243)|,\\
|[9]^9(15324)| &\neq |[9]^9(13542)|, \qquad (9.3)\\
|[9]^9(12435)| &\neq |[9]^9(14325)|.
\end{aligned}$$

S_9 is the set of permutations of $\{1, 2 \ldots, 9\}$, so the rest of the multisets μ of 9 letters in [9] have ≤ 8 distinct letters. Furthermore, when μ has ≤ 4 distinct letters, $|S_\mu(\tau)|$ is the same for any $\tau \in S_5$ since τ has 5 distinct letters. Thus, we only have to consider multisets μ of 9 letters in $[9] = \{1, 2, \ldots, 9\}$ which have exactly ℓ distinct letters for $5 \leq \ell \leq 8$. Let $w_\ell(k, n)$ be the number of n-letter words over $[k]$ which have exactly ℓ distinct letters. Then there is a $\binom{k}{\ell} : 1$ map $w_\ell(k, n)(\tau) \to [\ell]^n(\tau)$ induced by all $\binom{k}{\ell}$ order-preserving injections $[\ell] \hookrightarrow [k]$. Thus, $w_\ell(k, n)(\tau)$ divides $\binom{k}{\ell}$, so $\sum_{\ell=5}^{8} w_\ell(9, n)(\tau)$ divides $\gcd\left(\binom{9}{8}, \binom{9}{7}, \binom{9}{6}, \binom{9}{5}\right) = \gcd(9, 36, 84, 252) = 3$. On the other hand, $\sum_{\ell=1}^{8} w_\ell(9,n)(\tau = |[9]^9(\tau)| - |S_9(\tau)|$, so for any $\tau_1, \tau_2 \in S_5$,

$$|[9]^9(\tau_1)| - |[9]^9(\tau_2)| \equiv |S_9(\tau_1)| - |S_9(\tau_2)| \pmod 3$$

and hence, equations (9.2) imply inequalities (9.3). Therefore, we have

Corollary 9.2 *There are exactly 24 Wilf classes in S_5 for words avoiding permutations, each Wilf class containing exactly 1 West class.*

Corollaries 9.1 and 9.2 invite the following general conjecture.

Conjecture 9.3 *For any $\ell \in \mathbb{N}$, each Wilf class in S_ℓ (for words avoiding patterns) contains exactly 1 West class. In other words, West equivalence is the only nontrivial equivalence relation on S_ℓ (for words avoiding permutations) which implies Wilf equivalence.*

9.7 Further directions

First of all, we note that the sliding lemma needs to be proved in full generality for the Conjectures 9.1 and 9.2 to hold. Because of time constraints, we this proof for our future works. We should note that one alternative to this is proving the main lemma without the use of the sliding lemma. However, past attempts to avoid the use of the sliding lemma have not been successful so far.

Apart from this, there are several conjectures we can make either from our numerical data for small patterns as well as data in [27] and [32] or simply from our intuition, some of them analogous to earlier conjectures made for permutations avoiding permutations.

9.7.1 Monotonicity

CONJECTURE 9.4 (MONOTONICITY) *For any $k, \ell \in \mathbb{N}$ ($k \geq \ell$) and $\tau_1, \tau_2 \in S_\ell$, if $\exists n_0 \leq 2\ell - 1$ such that $|[k]^{n_0}(\tau_1)| < |[k]^{n_0}(\tau_2)|$, then $\forall n \geq n_0$, $|[k]^n(\tau_1)| < |[k]^n(\tau_2)|$, otherwise τ_1 and τ_2 are in the same Wilf class.*

The first part of this monotonicity conjecture for permutation patterns has been made earlier by West [27] (Question 3.4.2, p. 49), except for the bound on n_0.

DEFINITION 9.11 *If τ_1 and τ_2 are as in Conjecture 9.4, we say that τ_2 is asymptotically greater than τ_1 and write $\tau_1 \prec \tau_2$. In this case, τ_1 is said to be asymptotically smaller than τ_2.*

NOTATION 9.8 Note that, given Conjecture 9.4, "$\prec$" is indeed an order relation. We also write $\tau_1 \sim \tau_2$ if τ_1 and τ_2 are in the same Wilf class, and $\tau_1 \preceq \tau_2$ if $\tau_1 \prec \tau_2$ or $\tau_1 \sim \tau_2$. We can also use the same notation in case of permutations avoiding patterns with the corresponding change from $[k]^n$ to S_n in the definition (adding subscripts w for words and p for permutations when needed).

Unfortunately, an analogous monotonicity conjecture for all multisets of the same size and the same number of distinct elements is not true in general. For example, let $\mu_1 = \{1, 2, 2, 3, 4, 5, 6, 7\}$, $\mu_2 = \{1, 2, 3, 3, 4, 5, 6, 7\}$, then

$$|S_{\mu_1}(14523)| = 17583 < 17586 = S_{\mu_1}(14532)|,$$

$$|S_{\mu_2}(14523)| = 17599 > 17597 = S_{\mu_2}(14532)|.$$

Moreover, our data imply that such occurrences as above are fairly common in general. Even the conjecture that for any $\tau_1, \tau_2 \in S_\ell$, $\tau_1 \prec_w \tau_2 \Rightarrow \tau_1 \preceq_p \tau_2$, doesn't appear to be true as $|[8]^8(15243)| = 15,979,896 > 15,979,860 = |[8]^8(15324)|$, but $|S_9(15243)| = 260,202 < 260,204 = |S_9(15324)|$. The permutation patterns also have an additional symmetry which the rest of the word patterns don't, namely, one induced by the inverse operation, which is undefined for words.

REMARK 9.7 Note, however, that Conjecture 9.3 implies that $\tau_1 \sim_w \tau_2 \Rightarrow \tau_1 \sim_p \tau_2$.

9.7.2 Layered and semilayered patterns

Notice that the identity pattern is not the most avoided one either for $\ell = 4$ and $\ell = 5$. We conjecture that it never is and look at which patterns in S_ℓ are avoided by more words or by more permutations than $id_\ell \in S_\ell$.

DEFINITION 9.12 *A pattern $\tau \in S_\ell$ is called layered if it is a disjoint union of substrings (layers) such that the entries of τ are decreasing within each layer and increasing between layers (i.e. any entry in any layer is greater than any entry in any preceding layer). In other words, a permutation pattern is layered if it simultaneously avoids patterns* 231 *and* 312*.*

NOTATION 9.9 We will mark the layers within a layered pattern with hats, and denote the length of the layer s by ℓ_s.

REMARK 9.8 It is easy to see that all layered permutations are involutions, and hence, their symmetry classes are the same on words and on permutations. Also, for any layered permutation τ, $r(c(\tau)) = c(r(\tau))$ is also layered.

Remark 9.9 Notice that a layered permutation pattern is uniquely given by the vector of its layer lengths. For example, $(3, 2, 3, 1)$ corresponds to $\widehat{321}\widehat{54}\widehat{876}\hat{9}$, $(1, 1, \ldots, 1)$ [ℓ times] corresponds to the identity permutation id_ℓ. Since $\sum_s \ell_s = \ell$ and there are $2^{\ell-1}$ ordered partitions of ℓ as a sum of positive integers, we have that there are $2^{\ell-1}$ layered patterns in S_ℓ (much less than the total of $\ell!$ patterns in S_ℓ).

Remark 9.10 Layered permutations were treated in depth from a different perspective (packing layered patterns into permutations) in [15].

We conjecture that layered patterns are the most avoided ones, both by words and by permutations.

Conjecture 9.5 *Let $\ell \in \mathbb{N}$ and let $\tau_1, \tau_2 \in S_\ell$ be two patterns. Suppose that τ_1 is not Wilf-equivalent to any layered pattern while τ_2 is. Then*

$$\tau_1 \prec_w id_\ell \preceq_w \tau_2, \qquad \tau_1 \prec_p id_\ell \preceq_p \tau_2.$$

Furthermore, if neither id_ℓ nor its reversal is West-equivalent to any pattern in $\overline{\tau_2}$, then $id_\ell \prec_w \tau_2$ and $id_\ell \prec_p \tau_2$.

Remark 9.11 Notice that if τ_1 has ≤ 2 layers, we have $\tau_1 \sim id_\ell$ by West's conjecture.

In fact, we suspect that a stronger result holds on permutations for patterns not Wilf-equivalent to any layered pattern.

Conjecture 9.6 *If $\tau \in S_\ell$ is not Wilf-equivalent to a layered pattern, then*

$$\lim_{n\to\infty} |S_n(\tau)|^{\frac{1}{n}} < \lim_{n\to\infty} |S_n(id_\ell)|^{\frac{1}{n}} = (\ell-1)^2.$$

The truth of that conjecture is, of course, subject to the truth of the Stanley-Wilf conjecture (Conjecture 1.1 on page 20).

If we were to look for the "nicest" patterns among permutations, layered patterns would seem to be the ones. In particular, we conjecture that monotonicity for layered patterns holds uniformly in the following sense.

Conjecture 9.7 *Let $\ell \in \mathbb{N}$ and let $\tau_1, \tau_2 \in S_\ell$ be two layered patterns such that $\tau_1 \prec \tau_2$ and let n_0 be as in Conjecture 9.4. Then for any $n \geq n_0$ and any multiset μ of size n on $\geq \ell$ (distinct) letters, $|S_\mu(\tau_1)| \leq |S_\mu(\tau_2)|$.*

This conjecture should follow from the proof of another conjecture which applies to a wider class of patterns, which we call semilayered patterns.

Definition 9.13 *A pattern $\tau \in S_\ell$ is called* semilayered *if it is a disjoint union of layers such that the entries of τ are increasing between layers.*

Obviously, layered patterns are semilayered. We see that West's Conjecture (Conjecture 9.1) as well as our Conjecture 9.2 say that semilayered patterns of specific type are Wilf-equivalent. Now we add the following conjecture about asymptotical comparisons between certain semilayered patterns.

CONJECTURE 9.8 *Let $\ell \in \mathbb{N}$ and let $\tau_1, \tau_2 \in S_\ell$ be two semilayered patterns such that*

$$M_{\tau_1} = \begin{bmatrix} A & 0 & 0 \\ 0 & I_t & 0 \\ 0 & 0 & B \end{bmatrix}, \quad M_{\tau_2} = \begin{bmatrix} A & 0 & 0 \\ 0 & J_t & 0 \\ 0 & 0 & B \end{bmatrix} \tag{9.4}$$

for some nonempty patterns A and B. Then $\tau_1 \prec \tau_2$.

By the same reduction-to-boards argument as in section 9.4, pp. 86-88, this conjecture is a corollary of the following one.

CONJECTURE 9.9 *For any $t \in \mathbb{N}$, any $(0,1)$ multiset skew board λ_1/λ_2 (i.e. a $(0,1)$-skew board having at least one 1 in each row and exactly one 1 in each column) and any content γ,*

$$|S_{\lambda_1/\lambda_2,\gamma}(I_t)| \leq |S_{\lambda_1/\lambda_2,\gamma}(J_t)|,$$

and there exist sufficiently large skew boards λ_1/λ_2 (such that λ_1 is not a *rectangle and λ_2 is nonempty) and contents γ for which the inequality is strict.*

Finally, there is the question of which (layered) pattern τ^*_ℓ in S_ℓ is the most avoided one (i.e. the asymptotically greatest one).

CONJECTURE 9.10 (KÉZDY, SNEVILY, [20]) *If $\ell \in \mathbb{N}$ is even, say $\ell = 2l$, then the most avoided pattern (by permutations) is the one with layer length vector $(1, \underbrace{2, 2, \ldots, 2}_{l-1}, 1)$, i.e. $\hat{1}\widehat{32}\widehat{54}\ldots\widehat{(\ell-1)(\ell-2)}\hat{\ell}$.*

We conjecture that the same pattern is the most avoided one by words as well. When ℓ is odd, we distinguish two cases.

CONJECTURE 9.11 *If $\ell = 4l + 1$, $l \geq 1$ then the most avoided pattern has layer length vector $(1, \underbrace{2, \ldots, 2}_{l-1}, 3, \underbrace{2, \ldots, 2}_{l-1}, 1)$.*

CONJECTURE 9.12 *If $\ell = 4l + 3$, then the most avoided pattern has layer length vector $(1, \underbrace{2, \ldots, 2}_{l}, 1, \underbrace{2, \ldots, 2}_{l}, 1)$.*

In [5], it is shown that $(\forall n \in \mathbb{N})\,(\exists c)\,(\forall \tau \in S_n)\,(|S_n(\tau)| < c^n)$. Proving the last three conjectures, together with conjecture 9.7, may be a step to finding or at least bounding $c(\ell)$ for any ℓ. Also, Conjecture 9.5 and the result of [5] together imply the weak form of Stanley-Wilf conjecture (Conjecture 1.1 on page 20), i.e. that $|S_n(\tau)| < c^n$ for any pattern τ.

The τ_ℓ^*'s for $2 \leq \ell \leq 9$ (conjectured for $5 \leq \ell \leq 9$) are given in Table 9.3 on page 156.

τ	14325^a	12435^a	12345	14532
$\lvert[8]^8(\tau)\rvert$	15,983,073	15,983,073	15,983,028	15,981,453
τ	12453	15342	12534	15234
$\lvert[8]^8(\tau)\rvert$	15,981,426	15,981,399	15,981,390	15,981,363
τ	15423	13452	14523	13425
$\lvert[8]^8(\tau)\rvert$	15,981,345	15,981,327	15,981,300	15,980,004
τ	14352^b	15243^b	15324^c	13542^c
$\lvert[8]^8(\tau)\rvert$	15,979,896	15,979,896	15,979,860	15,979,860
τ	25143	13524	14253	24513
$\lvert[8]^8(\tau)\rvert$	15,979,737	15,978,774	15,978,630	15,978,469
τ	23514	25413	24153	25314
$\lvert[8]^8(\tau)\rvert$	15,978,226	15,978,214	15,976,992	15,976,740

Table 9.2: Cardinalities of $[k]^n(\tau)$ for $k = n = 8$, $\tau \in S_5$

.

ℓ	2	3	4	5	6	7	8	9
τ^*_ℓ	$\hat{1}\hat{2}$	$\hat{1}\hat{2}\hat{3}$	$\hat{1}\widehat{32}\hat{4}$	$\hat{1}\widehat{432}\hat{5}$	$\hat{1}\widehat{32}\widehat{54}\hat{6}$	$\hat{1}\widehat{32}\hat{4}\widehat{65}\hat{7}$	$\hat{1}\widehat{32}\widehat{54}\widehat{76}\hat{8}$	$\hat{1}\widehat{32}\widehat{654}\widehat{87}\hat{9}$

Table 9.3: The most avoided (layered) permutation patterns

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