ON SOME PROPERTIES OF PERMUTATION TABLEAUX

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Abstract. We consider the relation between various permutation statistics and properties of permutation tableaux. We answer some of the open problems of Steingrímsson and Williams [8], in particular, on the distribution of the bistatistic of numbers of rows and essential ones in permutation tableaux. We also consider and enumerate sets of permutation tableaux related to some pattern restrictions on permutations.

1. Introduction

Permutation tableaux are a particular class of ⅃-diagrams that were studied by Postnikov [6] and enumerated by Williams [9]. They are defined as follows. Given a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0)$ of an integer $m = \sum_i \lambda_i$, its *Young diagram* Y_λ of shape λ is a left-justified diagram of m boxes, with λ_i boxes in row i.

A *permutation tableau* $\mathcal{T}_n^k$, which we will also call a *1-hinge tableau*, is a partition λ whose Young diagram Y_λ is contained in a $k \times (n-k)$ rectangle aligned with its top and left edges, together with a filling of the cells of Y_λ with 0s and 1s that satisfies the following properties:

- **(column):** Each column of the rectangle contains at least one 1.
- **(1-hinge):** A cell in Y_λ with a 1 above it in the same column and a 1 left of it in the same row must contain a 1.

The filling satisfying the column and hinge properties is called *valid*. The 1s in $\mathcal{T}_n^k$ that are topmost in their columns or leftmost in their rows are called *essential*, and the remaining 1s, i.e. those that are forced by the 1-hinge property, are called *induced*. Note that the column property implies that Y_λ must have exactly $n-k$ columns and at most k rows. Note also that some rows of $\mathcal{T}_n^k$ (as opposed to columns) may contain all zeros. Removing the column requirement yields the definition of Postnikov's ⅃("Le")-tableaux [6] that are used to describe and enumerate totally positive Grassmann cells.

Alternatively, a permutation tableau $\mathcal{T}_n^k$ may be thought of as a filling of a $k \times (n-k)$ rectangle with 0s, 1s and 2s such that the cells inside Y_λ are filled with 0s and 1s so as to satisfy the column and 1-hinge properties, and the cells outside Y_λ are filled with 2s.

Properties of permutation tableaux were studied by Steingrímsson and Williams [8]. They gave a simpler description of a map described by Postnikov [7] that takes permutation tableaux contained in a $k \times (n-k)$ rectangle to permutations in $\mathfrak{S}_n$ with k weak excedances, and proved that this map Φ is a bijection that also preserves many other statistics. The bijection Φ will be described in the next section.

One of the conjectures made in [8, Section 7] is that the distribution of permutation tableaux according to the number of essential 1s is equal to that of the number of cycles in

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permutations, i.e. it is given by the signless Stirling numbers of the first kind. Moreover, [8] conjectured that the joint distribution of tableaux according to the number of rows and the semiperimeter minus number of essential 1s equals that of permutations according to the number of weak excedances and the number of cycles of a permutation, when written in standard cycle form (which is the same distribution as that of permutations according to descents and left-to-right minima). In this paper we will give a simple natural bijection on ⅃-tableaux that induces the conjectured bijection above and preserves several other statistics.

Another bijection Ψ on $\mathfrak{S}_n$ defined in [8] translates certain statistics on permutations corresponding to entries of their permutation tableaux (determined by Φ) into certain linear combinations of generalized permutation patterns. In particular, Ψ maps permutations avoiding generalized pattern 2-31 to permutation tableaux with exactly one 1 in each column. Since the number of permutations in $\mathfrak{S}_n$ avoiding 2-31 is C_n, the nth Catalan number, we give a simple description of permutations whose tableau has a single 1 in each column in terms of noncrossing partitions. Also, as simple applications of the Φ and Ψ, we get the properties of the tableaux of permutations restricted by some 3-letter patterns.

The outline of this paper is as follows. In Section 2 we simplify the presentation of the map from permutations to permutation tableaux given in [8] as well as present solution to Open Problem 5 of [8]. In Section 3, we define generalized permutation patterns and present some results that link pattern occurrences in permutations to various properties and statistics on the associated tableaux. We also solve Open Problems 4 and 6 of [8], in part by using the results of Section 2. Finally, in Section 4, we define the essential 1s and doubly essential 1s of a tableau and present the main result of the paper: the joint distribution of the number of doubly essential 1s plus rows with no 1s and the number of rows over permutation tableaux is the same as the joint distribution of the number of cycles and the number of weak excedances over permutations. This solves part of Open Problem 3 of [8].

2. Permutations and permutation tableaux

Here we briefly describe a bijection Φ from [8]. We also give a different proof that Φ is a bijection, essentially recovering the tableaux from the corresponding permutation by reconstructing the columns of the tableau from left to right (or, similarly, rows from top to bottom), as opposed to the right-to-left column construction in [8] (or the similar bottom-to-top row construction).

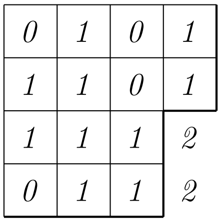
Given a permutation tableau $\mathcal{T}_n^k$, its *diagram* $D(\mathcal{T}_n^k)$ is defined as follows. The southeast border of a Young diagram Y_λ gives a path $P = (P_i)_{i=1}^n$ of length n from the northeast corner of the $k \times (n-k)$ rectangle containing Y_λ to the southwestern corner of that rectangle. Label each step P_i in P with i, for $i \in [n]$ (where $[n] = \{1, 2, \ldots, n\}$). Now, given an edge P_i, also label with i the edge Q_i on the opposite end of the row (if P_i is vertical) or column (if P_i is horizontal) containing the edge P_i. Replace each 1 in $\mathcal{T}_n^k$ with a vertex and delete all 0s. From each vertex draw edges east and south either to the closest vertices in the same row and the same column or to the labels i of some edge P_i in P, if there are no more vertices in that direction. The resulting picture is the diagram $D(\mathcal{T}_n^k)$. It is also convenient to consider $D(\mathcal{T}_n^k)$ together with the edges from the labels i of edges Q_i on the northwestern boundary of $\mathcal{T}_n^k$ to the closest (leftmost) vertex in the same row (if Q_i is vertical) or the closest (rightmost) vertex in the same column (if Q_i is horizontal). We will denote the resulting diagram $D'(\mathcal{T}_n^k)$ and call it the *expanded diagram* of $\mathcal{T}_n^k$. It is also convenient to think of edge labels $i \in [n]$

as labeling a row or a column between P_i and Q_i, and label each cell in the tableau by the ordered pair of its row and column labels.

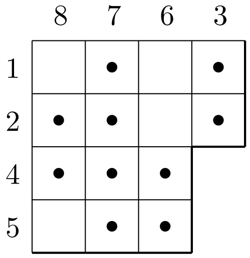
Given a tableau $\mathcal{T}_n^k$ as above, the permutation $\pi = \Phi(\mathcal{T}_n^k)$ as defined as follows. For each $i \in [n]$, consider a zigzag path in $D'(\mathcal{T}_n^k)$ that starts at Q_i (going south or east depending on whether Q_i is horizontal or vertical) and switchings direction between south and east at every vertex it encounters. If that path terminates at P_j, then we set $\pi(i) = j$. Alternatively, in $D(\mathcal{T}_n^k)$ we replace the first edge of the path starting from Q_i by an edge north or west from P_i to the vertex in row or column i that is farthest from P_i (and closest to Q_i).

Example 2.1.

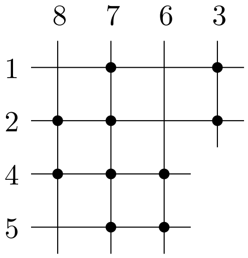
$\mathcal{T} = \mathcal{T}_8^4 =$



or



$D'(\mathcal{T}) =$



Following the southeast paths in $D'(\mathcal{T})$ starting with each letter from 1 to $n = 8$ and switching direction at each dot in $D'(\mathcal{T})$, we see that tableau $\mathcal{T}$ corresponds to the permutation $\pi = \Phi(\mathcal{T}) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 1 & 8 & 7 & 4 & 2 & 5 \end{pmatrix} \in \mathfrak{S}_8$. Note also that $k = 4$ is the number of rows of $\mathcal{T}$.

It is shown in [8] that Φ is a bijection, $\pi(i) \geq i$ if P_i is vertical and $\pi(i) < i$ if P_i is horizontal, hence $\pi = \Phi(\mathcal{T}_n^k)$ has k *weak excedances* (positions i such that $\pi(i) \geq i$) and $n - k$ *deficiencies* (positions i such that $\pi(i) < i$). In fact, i is a fixed point of π (i.e. $\pi(i) = i$) if and only if the row of $\mathcal{T}_n^k$ labeled i does not contain a 1. We will refer to Φ as the standard bijection from permutation tableaux to permutations and refer to $\Phi(\mathcal{T})$ as the permutation of $\mathcal{T}$ and to $\Phi^{-1}(\pi)$ as the tableau of π.

In other words, π is a derangement if and only if every row of $\mathcal{T} = \Phi^{-1}(\pi)$ also contains a 1. In that case, we may reflect $\mathcal{T}$ across the NW-SE diagonal and obtain another tableau $\mathrm{refl}(\mathcal{T})$ with the same properties. Moreover, each southeast path $Q_i(\mathcal{T}) \to P_{\pi(i)}(\mathcal{T})$ is thus reflected onto a southeast path $Q_{n+1-i}(\mathrm{refl}(\mathcal{T})) \to P_{n+1-\pi(i)}(\mathrm{refl}(\mathcal{T}))$. Thus, if $\pi = \Phi(\mathcal{T})$ and $\sigma = \Phi(\mathrm{refl}(\mathcal{T}))$, then $\sigma(n+1-i) = n+1-\pi(i)$ for all $i \in [n]$, i.e. σ is the reversal of the complement of π. This answers Open Problem 5 of [8] that asked which operation on derangements corresponds to the reflection of their associated tableaux across the main (northwest-southeast) diagonal.

Remark 2.2. Note that each cell with a 0 or 1 in $\mathcal{T}$ is a common point of two southeast paths in $D'(\mathcal{T})$ that are uniquely determined by the cell, and these paths cross if the cell contains a 0 and touch, but do not cross, if the cell contains a 1.

We will need the following result in later sections.

Lemma 2.3. *For any tableau $\mathcal{T}$, any two southeast paths in $D'(\mathcal{T})$ cross at most once, and that intersection must be in the first leg of at least one of the paths (i.e. before reaching the first 1). Thus, each 0 in $\mathcal{T}$ corresponds to a unique pair of crossing paths in $D'(\mathcal{T})$.*

Proof. Suppose that two paths have a common cell c that is in neither of their respective first legs. Then one of the paths approaches it from a 1 higher in the same column, and the other path approaches it from a 1 to the left in the same row. Hence, by the 1-hinge rule the cell c also must contain a 1, so these paths can only touch, but not intersect, at c. ☐

Note that each of the two intersecting paths may correspond to an excedance (E) or a non-excedance (N). Suppose that two southeast paths cross at a cell c labelled (i, j) (i.e. in row labelled i and column labelled j). Then $i < j$. Let p_{row} and p_{col} be the paths entering c from the left and from above, respectively.

If p_{row} starts with a south edge and p_{col} starts with an east edge then c does not lie on either of their first leg, so these paths cannot cross. Thus, either p_{row} starts with an east edge or p_{col} starts with a south edge, so we may have three types of southeast path intersections in $D'(\mathcal{T})$: EE, NN, EN. Let $\text{EE}(\mathcal{T})$, $\text{NN}(\mathcal{T})$, $\text{EN}(\mathcal{T})$ be the sets of corresponding southeast path intersections in $\mathcal{T}$.

If we have an EN intersection, then p_{col} starts south at the column labelled $j_0 \geq j$ and p_{row} starts east at the row labelled $i_0 \leq i$. Then $i_0 \leq i < j \leq j_0$ (and either $i = i_0$ or $j = j_0$), so $i_0 < j_0$.

Cells (i_0, j_0) with $i_0 > j_0$ that are filled with 2s correspond to NE pairs of southeast paths p_{row} starting at row labelled i_0 and p_{col} starting at column labelled j_0, which can never touch or cross (see [8]). We denote the set of these cells by $\text{NE}(\mathcal{T})$.

Example 2.4.

	8	7	6	3
1	EN	•	NN	•
2	•	•	EN	•
4	•	•	•	NE
5	EE	•	•	NE

Several permutation statistics related to corresponding tableaux were also introduced in [5] and used in [8]. These statistics essentially count all possible types of pairs of southeast paths $(i \mapsto \pi(i),\ j \mapsto \pi(j))$ in the tableau corresponding to a given permutation.

$$\begin{aligned} A_{EE}(\pi) &= |\{(i,j) \mid j < i \leq \pi(i) < \pi(j)\}| \\ A_{NN}(\pi) &= |\{(i,j) \mid \pi(j) < \pi(i) < i < j\}| \\ A_{EN}(\pi) &= |\{(i,j) \mid j \leq \pi(j) < \pi(i) < i\}| \\ A_{NE}(\pi) &= |\{(i,j) \mid \pi(i) < i < j \leq \pi(j)\}| \\ C_{EE}(\pi) &= |\{(i,j) \mid j < i \leq \pi(j) < \pi(i)\}| \\ C_{NN}(\pi) &= |\{(i,j) \mid \pi(i) < \pi(j) < i < j\}| \end{aligned} \tag{2.1}$$

Then it is easy to see that

$$\begin{aligned} A_{EE}(\pi) &= |\text{EE}(\mathcal{T})| \\ A_{NN}(\pi) &= |\text{NN}(\mathcal{T})| \\ A_{EN}(\pi) &= |\text{EN}(\mathcal{T})| \\ A_{NE}(\pi) &= |\text{NE}(\mathcal{T})| = \#2\text{s}(\mathcal{T}), \end{aligned} \tag{2.2}$$

so the discussion above gives a direct bijective proof that

$$A_{EE}(\pi) + A_{NN}(\pi) + A_{EN}(\pi) = \#0\text{s}(\mathcal{T}), \tag{2.3}$$

which was shown in [8] by a more complicated argument. We also note that [8] shows that

$$C_{EE}(\pi) + C_{NN}(\pi) = \#\text{nontop 1s}(\mathcal{T}).$$

Now we will introduce a bit of terminology.

Definition 2.5. If $\pi(i) \geq i$ is a weak excedance (*wex*), we will call $\pi(i)$ a *weak excedance top (wex top)* of π, and call i a *weak excedance bottom (wex bottom)* of π. If $\pi(i) < i$, we will call $\pi(i)$ a *non-weak-excedance bottom (nonwex bottom)* of π, and call i a *non-weak-excedance top (nonwex top)* of π.

We will now describe a way to recover $\mathcal{T}_n^k$ from $\pi = \Phi(\mathcal{T}_n^k) \in \mathfrak{S}_n$ starting from the leftmost column. Let m be the largest position of a non-fixed point in π. Then $\pi(m) < m$, and $\pi(i) = i$ for all $i > m$. Since it is trivial to recover the edges P_i for $i > m$, we may assume that without loss of generality that $m = n$ (so P_n is a horizontal edge).

Theorem 2.6. *Assume that $\pi \in \mathfrak{S}_n$ is such that $\pi(n) < n$, and let $\mathcal{T} = \Phi^{-1}(\pi)$. Suppose that the dots in the leftmost column of $\mathcal{T}$ are in rows labeled $i_1 < i_2 < \cdots < i_r$. Furthermore, let $\mathcal{T}'$ be the tableau obtained by removing the leftmost column of $\mathcal{T}$ and replacing P_n with a vertical edge (so that n becomes a fixed point), and let $\pi' = \Phi(\mathcal{T}')$. Then $\pi = \pi' \circ (i_1\, i_2\, \ldots\, i_r\, n)$, $\pi(n) < \pi(i_1) < \cdots < \pi(i_r)$ and $\pi(i_1) < \cdots < \pi(i_r)$ are the successive non-fixed-point left-to-right maxima of the subsequence of π consisting of values greater than $\pi(n)$.*

Proof. For $1 \leq j < r$, the first three steps of the southeast path from Q_{i_j} to $P_{\pi(i_j)}$ are east from Q_{i_j} to cell (i_j, n), then south to (i_{j+1}, n), then east from (i_{j+1}, n). Thus, $\pi(i_j) = \pi'(i_{j+1})$. Similarly, the path starting from Q_{i_r} goes east to (i_r, n), then south to P_n, so $\pi(i_r) = n = \pi'(n)$. Likewise, the path from Q_n starts south to (i_1, n), then turns east, so $\pi(n) = \pi'(i_1)$. Thus, $\pi = \pi' \circ (i_1\, i_2\, \ldots\, i_r\, n)$ as claimed.

Note that paths $p(i_j) : i_j \to \pi(i_j)$ and $p(i_{j+1}) : i_{j+1} \to \pi(i_{j+1})$ meet at a vertex in cell (i_{j+1}, n), where $p(i_j)$ enters it traveling south and leaves east while $p(i_{j+1})$ enters it traveling east and leaves south. Hence, we can see by induction that at each row $p(i_{j+1})$ is to the east of $p(i_j)$ and at each column $p(i_j)$ is to the south of $p(i_{j+1})$, so if $p(i_j)$ and $p(i_{j+1})$ meet again at a cell it must contain an induced 1, so they cannot cross. It follows from that $\pi(i_j) < \pi(i_{j+1})$ for all $j < r$, and similarly that $\pi(n) < \pi(i_1)$.

Now let $\ell \in [n]$ be a row label such that $i_{j-1} < \ell < i_j$ (if $j = 1$, we simply let $\ell < i_1$). Then the path from Q_ℓ starts east and either continues to P_ℓ or first turns south at a cell (ℓ, m) for some $m < n$. Since (ℓ, m) (if it exists) is northeast of (i_{j+1}, n), the same argument as before applies to show that $\pi(i_j) > \pi(\ell)$. Therefore, $\pi(i_{j+1})$ is the leftmost value to the right of $\pi(i_j)$ that is greater than $\pi(i_j)$, so the theorem follows. □

Note that the largest entry n in the (increasing) cycle above gives the label of the leftmost column, while the remaining entries give the labels of the rows containing dots in that column. Iterating the operation yields all the cells in $\mathcal{T}$ containing dots. The largest elements in the cycles are column labels, the rest are row labels.

Example 2.7. Let $\pi = 36187425$. Then we have

i	1	2	3	4	5	6	7	8	cycle
π	3	**6**	1	**8**	7	4	2	**5**	(248)
π'	**3**	**5**	1	**6**	**7**	4	**2**	8	(12457)
π''	2	3	1	**5**	**6**	**4**	7	8	(456)
π'''	**2**	**3**	**1**	4	5	6	7	8	(123)
ϵ	1	2	3	4	5	6	7	8	

$$\mathcal{T} = \begin{array}{c|c|c|c|c|} & 8 & 7 & 6 & 3 \\ \hline 1 & & \bullet & & \bullet \\ \hline 2 & \bullet & \bullet & & \bullet \\ \hline 4 & \bullet & \bullet & \bullet \\ \cline{1-4} 5 & & \bullet & \bullet \\ \cline{1-4} \end{array}$$

so $\pi = (123)(456)(12457)(248)$, and the path $P(\pi) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ v & v & h & v & v & h & h & h \end{pmatrix}$, where v and h denote the vertical and horizontal edges, respectively, so $\mathcal{T} = \Phi^{-1}(\pi)$ has dots in cells labeled $(1,3)$, $(2,3)$, $(4,6)$, $(5,6)$, $(1,7)$, $(2,7)$, $(4,7)$, $(5,7)$, $(2,8)$, $(4,8)$. Note that row labels increase from top to bottom, while column labels increase from right to left.

Remark 2.8. Note that the column rule implies that there are no 1-cycles in this decomposition of π. Likewise, the 1-hinge rule implies that if a cycle c in the product as above contains an element i, and another cycle c' to its left contains elements j_1, j_2 such that $j_1 < i < j_2$, then c' also contains i.

Definition 2.9. We call the (unique) representation of a permutation π as a product of increasing cycles subject to the conditions in Remark 2.8 the *column decomposition* of π.

Likewise, it is easy to see that we can determine $\mathcal{T}$ by rows from top to bottom by decomposing $\mathcal{T}$ as a product of *decreasing* cycles. In this case, however, we may have 1-cycles, which will correspond to rows without dots. Here, at each step, we will need to find positions of the successive non-fixed-point *right-to-left minima* of the subsequence of π consisting of values smaller than $\pi(s)$, where s is the position of the leftmost non-fixed point. After arriving at the identity permutation, we add the remaining elements as fixed points.

Example 2.10. Given $\pi = 36187425$, we obtain $\pi = (765)(8764)(8732)(731)$ similarly to Example 2.7.

Remark 2.11. As in Remark 2.8, note that the 1-hinge rule implies that if an element i occurs in some cycle c in the product as above, and another cycle c' to the left of c contains elements j_1, j_2 such that $j_1 > i > j_2$, then c' also contains i.

Definition 2.12. We call the (unique) representation of a permutation π as a product of decreasing cycles subject to the conditions in Remark 2.11 the *row decomposition* of π.

3. TABLEAUX AND PERMUTATION PATTERNS

Now we need to define the notion of a pattern. A *(classical) permutation pattern* is an order-isomorphism type of a sequence of totally ordered letters. Such a sequence is then referred to as an *occurrence* or *instance* of that pattern. For example, $\sigma = 214653$ contains instances of pattern $\pi = 2\text{-}3\text{-}1$ at subsequences 463 and 453 but no instance of pattern $\tau = 3\text{-}1\text{-}2$. In this case, we say that σ *contains* π and *avoids* τ. The dashes, which are often dropped when referring to classical patterns, are used to indicate that the terms involved in an occurrence of the pattern may be separated by an arbitrary number of other terms. A *generalized permutation pattern* (introduced by [1]) is a pattern where some letters adjacent in a pattern must also be adjacent in a containing permutation. Such adjacent pairs in a pattern are then *not* separated by a dash. For example, 453 is an instance of pattern 2-31 in 214653, while 463 is an instance of pattern 23-1, but not 2-31. If π is a pattern and σ is a permutation, we write $(\pi)\sigma$ for the number of occurrences of π in σ. We also write $\mathfrak{S}_n(\pi) = \{\sigma \in \mathfrak{S}_n \mid (\pi)\sigma = 0\}$. One of the earliest results concerning patterns is that $|\mathfrak{S}_n(\pi)| = C_n$, the nth Catalan number, for any classical pattern $\pi \in \mathfrak{S}_3$. The same holds for $\pi =$ 2-31 and $\pi =$ 31-2 since $\mathfrak{S}_n(\text{2-31}) = \mathfrak{S}_n(\text{2-3-1})$ and $\mathfrak{S}_n(\text{31-2}) = \mathfrak{S}_n(\text{3-1-2})$. See [2] for more on patterns and pattern avoidance.

We will now show a nice application of row decomposition. In [8], another bijection $\Psi : \mathfrak{S}_n \to \mathfrak{S}_n$ is given that implies that the permutation tableaux containing a single 1 in each column (i.e. the fewest possible number of 1s) are the images of permutations avoiding pattern 2-31, and thus are counted by the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Open Problem 6 in [8] asks for "a bijection from these tableaux to any well-known set of objects enumerated by Catalan numbers". We will give such a bijection using Remark 2.11.

Theorem 3.1. *There is a natural bijection between k-row permutation tableaux of semiperimeter n with $n-k$ 1s (i.e. a single 1 per column) and noncrossing partitions of $[n]$ with k blocks.*

Hence (see [9]), the number of these tableaux with semiperimeter n and k rows is the number of partitions of $[n]$ with k blocks, i.e. the Narayana number $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.

Proof. Suppose that $\mathcal{T}$ is a tableau with a single 1 in each column, and let $\pi = \Phi(\mathcal{T}) \in \mathfrak{S}_n$. Consider the row decomposition of π. Since each column contains a single 1, each column label occurs in only one of the cycles. Each row label also occurs in a single cycle. Thus, every label occurs once in the row decomposition of π, hence the cycles in the row decomposition of π are mutually disjoint, and the underlying sets for these cycles form a set partition Π of $[n]$. Suppose that there are cycles $\gamma_1 \neq \gamma_2$ in the row decomposition of π such that γ_1 contains elements a, c and γ_2 contains elements b, d such that $a > b > c > d$. Thus, if γ_2 is to the left of γ_1, then γ_2 also contains b, and if γ_1 is to the left of γ_2, then γ_1 also contains c. This contradicts the fact that each element in $[n]$ must occur in a single cycle of the row decomposition of π, so Π is noncrossing. ☐

We note that the set E_n of permutations whose cycle decomposition corresponds to noncrossing partitions of n occurs in [3] as the set of sequences of halves of even-valued entries of 3-1-4-2 avoiding Dumont permutations of the second kind. Note also that E_n is exactly the set of permutations whose column decomposition contains only 2-cycles.

The next theorem describes the tableaux of 3-2-1 avoiding permutations.

Theorem 3.2. *The tableaux of* 3-2-1 *avoiding permutations are exactly those whose rows and columns are all nondecreasing from left to right and from top to bottom, respectively.*

Proof. Note that π is a 3-2-1 avoiding permutation if and only if each element of π is either a left-to-right maximum or a right-to-left-minimum, i.e. if and only if π is the identity or a union of two nondecreasing subsequences. Again, without loss of generality assume that n is not a fixed point of π, i.e. $\pi(n) < n$. Suppose that the first column contains a 0 at row ℓ that is underneath a 1 at row i. Then as in the proof of Theorem 2.6, we have $i < \ell \leq \pi(\ell) < \pi(i)$. Hence, to avoid an occurrence of 3-2-1 in π, we must have $\pi(m) > \pi(\ell) \geq \ell$ for all $m > \ell$, so π must have at least $n-\ell+1$ values greater than $\pi(\ell) \geq \ell$ ($\pi(i)$ and all $\pi(m)$ for $m > \ell$), which is impossible. Therefore, the leftmost column of π must have all 0s atop all 1s. Moreover, if the rows containing a dot in the leftmost column are labeled i_j $(1 \leq j \leq r)$, then the sequence $\{\pi(i_j)\}_{j=1}^{r}$, is increasing.

Let π' be the permutation defined as in Theorem 2.6. Assume that the sequence of wex tops of π' (see Definition 2.5) is increasing. We have $\pi(\ell) = \pi'(\ell)$ for a wex bottom $\ell < i_1$, as well as $\pi(i_j) = \pi'(i_{j+1})$ for $1 \leq j < r$, and $\pi(i_r) = \pi'(n) = n$, so the sequence of wex tops of π is increasing if the sequence of wex tops of π' is increasing. Since the identity tableau has no columns and increasing wex tops, we see by induction that if each column of π has all 0s atop all 1s, then the sequence of wex tops of π is increasing.

Thus, if π is 3-2-1 avoiding then the sequence of wex tops of π is increasing and each column of the tableau of π has all 0s atop of all 1s. Likewise, π is the sequence of nonwex bottoms of π is increasing and each row of the tableau of π has all 0s to the left of all 1s. Conversely, if both wex tops and nonwex bottoms are increasing, then π is a union of at most 2 subsequences and hence avoids 3-2-1. $\square$

Note that this means that the number of monotone tableaux with semiperimeter n and k rows is the number of permutations in $\mathfrak{S}_n$(3-2-1) with k weak excedances, i.e. again the Narayana number $N(n, k)$.

In [8], a bijection Ψ on $\mathfrak{S}_n$ is given that translates certain pattern statistics on permutations into *alignment* and *crossing* statistics (see [8]) of the tableaux of their images. We will not need the description of the bijection itself here, only some of its properties.

Given a pattern τ, recall that $(\tau)\sigma$ is the number of occurrences of τ in σ. Also, let $\operatorname{des}\sigma$ be the number of descents of σ, and let $\operatorname{wex}\pi$ be the number of weak excedances of π. From [8, Theorem 24] we have that, for $\pi = \Psi(\sigma)$,

$$\begin{aligned}
\operatorname{des}\sigma &= \operatorname{wex}\pi - 1 \\
(\text{31-2})\sigma &= A_{EE}(\pi) + A_{NN}(\pi) \\
(\text{21-3})\sigma + (\text{3-21})\sigma - \binom{\operatorname{des}\sigma}{2} &= A_{EN}(\pi) \\
(\text{2-31})\sigma &= C_{EE}(\pi) + C_{NN}(\pi) \\
(\text{1-32})\sigma + (\text{32-1})\sigma - \binom{\operatorname{des}\sigma}{2} &= A_{NE}(\pi)
\end{aligned} \tag{3.1}$$

As proved in [8], this implies (via the reverse complement map applied to σ) that the following pairs of statistics are equidistributed on permutations:

(1) $A_{EE} + A_{NN}$ and $C_{EE} + C_{NN}$,
(2) A_{EN} and A_{NE}.

It is surprisingly easy to show that the statistics A_{EN} and A_{NE} are equidistributed. Let $\mathrm{irc} = \mathrm{i} \circ \mathrm{r} \circ \mathrm{c}$ be the bijection of inverse of reversal of complement (or reflection across the antidiagonal of the permutation diagram). In other words, for $\pi \in \mathfrak{S}_n$, $\mathrm{irc}(\pi)(i) = j$ if and only if $\pi(n+1-j) = n+1-i$. Then irc preserves wex, A_{EE}, A_{NN}, C_{EE}, C_{NN}, and exchanges A_{EN} and A_{NE}.

Moreover, Open Problem 4 of [8] asks if the set of 0s in a permutation tableau π can be partitioned into subsets that correspond to occurrences of (3-12) and (21-3) + (3-21), respectively, in $\sigma = \Psi^{-1}(\pi)$. Now (2.2) and (3.1) together clearly imply that the desired sets are exactly $\textsc{en}(\mathcal{T})$ and $\textsc{ee}(\mathcal{T}) \cup \textsc{nn}(\mathcal{T})$, where $\mathcal{T} = \Phi^{-1}(\pi)$ is the tableau of π.

Recall that [8] showed that $\Psi(\mathfrak{S}_n(2\text{-}31))$ is the set of permutations whose tableaux have a single 1 per column. Here we establish another pattern-related result concerning Ψ. Recall that $\mathfrak{S}_n(2\text{-}31) = \mathfrak{S}_n(2\text{-}3\text{-}1)$ and $\mathfrak{S}_n(31\text{-}2) = \mathfrak{S}_n(3\text{-}1\text{-}2)$.

Theorem 3.3. $\Psi(\mathfrak{S}_n(31\text{-}2)) = \mathfrak{S}_n(3\text{-}2\text{-}1)$, *i.e.* $\Psi(\mathfrak{S}_n(31\text{-}2))$ *is the set of permutations whose tableaux have nondecreasing rows and columns.*

Proof. Let $\pi \in \mathfrak{S}_n$, and let $\sigma = \Psi(\pi)$. It is shown in [8] that the occurrences of 3-12 in π correspond to pairs $i, j \in [n]$ such that $j < i \le \sigma(i) < \sigma(j)$ or $\sigma(j) < \sigma(i) < i < j$. In other words, i and j are both wex bottoms or both nonwex tops (i.e. $\sigma(i)$ and $\sigma(j)$ are both wex tops or both nonwex bottoms), and σ has an inversion at positions (i, j). Therefore, π avoids 31-2 if and only if the sequence of wex tops of σ and the sequence of nonwex bottoms of σ both have no inversions, i.e. are increasing. In other words, σ is a union of at most two increasing subsequences, that is $\sigma \in \mathfrak{S}_n(3\text{-}2\text{-}1)$. $\square$

Note that, while the irc map is very easy, the corresponding map $\mathrm{tirc} = \Phi^{-1} \circ \mathrm{irc} \circ \Phi$ on tableaux seems difficult to describe in general. However, in the special case of $\mathfrak{S}_n(3\text{-}2\text{-}1)$ and monotone tableaux, tirc becomes very easy as well. To apply tirc to a monotone tableau, simply rotate it by 180°, replace every label i with $n + 1 - i$ and rename all 0s as 2s and all 2s as 0s. It is interesting how this map generalizes to tirc on all permutation tableaux.

4. Essential 1s

Definition 4.1. Given a permutation tableau $\mathcal{T}$, we call the topmost 1 in each column and the leftmost 1 in each row an *essential* 1. If a 1 is both the leftmost 1 in its row and the topmost 1 in its column, then we call it *doubly essential.* Let $\mathrm{ess}(\mathcal{T})$ and $\mathrm{dess}(\mathcal{T})$ be the number of essential 1s and doubly essential 1s in $\mathcal{T}$ and let $\mathrm{no1rows}(\mathcal{T})$ be the number of rows in $\mathcal{T}$ that have no 1s.

Note that for any tableau $\mathcal{T}$ in a $k \times (n-k)$ rectangle, we have

$$\mathrm{ess}(\mathcal{T}) + \mathrm{dess}(\mathcal{T}) + \mathrm{no1rows}(\mathcal{T}) = k + (n-k) = n$$

by a simple sieve argument. Since each nonessential 1 is induced by a pair of essential 1s, each tableau is determined by its essential 1s. It was conjectured in the Open Problem 3 of [8] that the distribution of the bistatistic $(n - \mathrm{ess}, \mathrm{rows})$ of the number of essential 1s and the number of rows on permutation tableaux is the same as that of the bistatistic $(\mathrm{cycles}, \mathrm{wex})$ of the number of cycles and the number of weak excedances on permutations. We will solve Open Problem 3 by showing the following.

Theorem 4.2. *The bistatistic* $(\mathrm{dess} + \mathrm{no1rows}, \mathrm{rows})$ *on tableaux bounded by a path of length* n*, and the bistatistic* $(\mathrm{cycles}, \mathrm{wex})$ *on permutations in* $\mathfrak{S}_n$ *have the same distribution.*

Since each row with no 1s corresponds to a fixed point, it is enough to show that the theorem is true on derangements, so that the corresponding tableaux have a 1 in every row. In other words, we need to prove the following theorem.

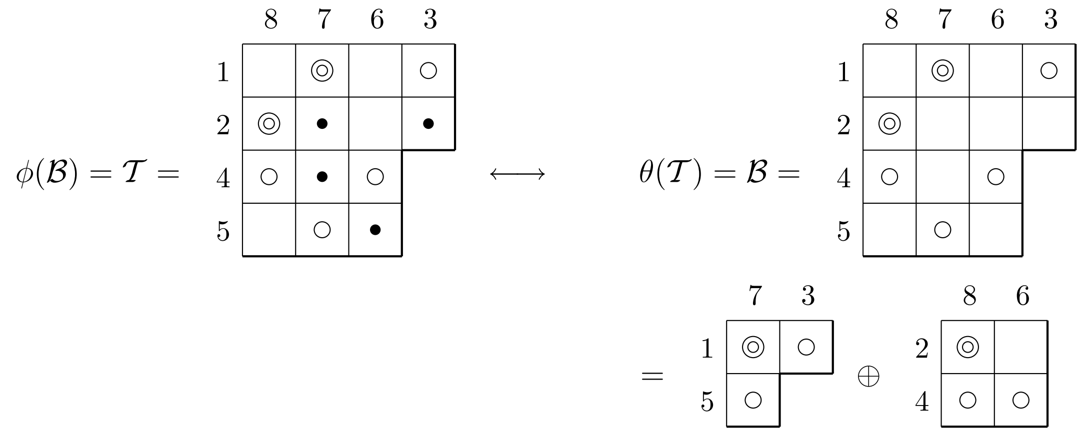
Theorem 4.3. *The bistatistic* (dess, rows) *on tableaux bounded by a path of length n that have a 1 in every row, and the bistatistic* (cycles, wex) *on derangements in $\mathfrak{S}_n$ have the same distribution.*

Consideration of essential 1s suggests a different type of tableau, which we will call a *bare tableau* or a *0-hinge tableau*, that results from replacing all nonessential 1s in a permutation tableau with 0s. A bare tableau is defined almost the same way as a permutation tableau, except that the 1-hinge property is replaced with the corresponding 0-hinge property:

> **(0-hinge):** A cell in Y_λ with a 1 above it in the same column and a 1 to its left in the same row must contain a 0.

We also define two maps θ and ϕ (see (4.1)) on tableaux of the same shape that result in removal and filling of all nonessential 1s, respectively, as well as a map Θ from bare tableaux to permutations, similar to Φ, i.e. given by southeast paths from the northwestern to the southeastern boundary of the tableau that switch direction at each 1. In the diagram below, empty circles denote essential 1s and double circles denote doubly essential 1s while black dots denote the induced 1s.

(4.1)



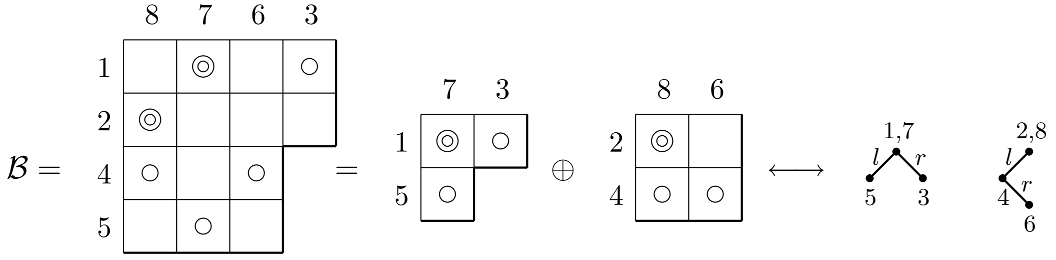
$$\Phi(\mathcal{T}) = 36187425 = (13)(264857) \quad \longleftrightarrow \quad \Theta(\mathcal{B}) = (1573)(2648) = 56187432$$

It is easy to see that the diagram $D(\mathcal{B})$ of a bare tableau is a binary forest, since every vertex may only have a single edge connecting it to a vertex above or to the left of it and at most two edges to vertices south and east of it. Indeed, if there are edges south and east to the same vertex, then $\mathcal{B}$ has a 1-hinge at that vertex, which is impossible. Moreover, the doubly essential 1s are exactly the roots of those binary trees.

The resulting trees are labeled as follows. The root has the two labels of the row and column of the cell that contains it. Each nonroot vertex has a single label: the column (resp. row) label of the cell containing it if that vertex is reached by an edge south (resp. east) when traveling from the root. Call a nonroot vertex a *left* son if it gets a row label, and a *right* son if it gets a column label. Then the following property is easy to see: each left son has the least label in its subtree, and each right son has the greatest label in its subtree. Also, the root gets the least and the greatest label in each tree, and thus has the properties of both a left son and a right son.

Thus, each bare tableau $\mathcal{B}$ can be decomposed into several tableaux each of which corresponds to a single binary tree in $D(\mathcal{B})$ labeled as described above.

(4.2)



The permutation $\tau = \Theta(\mathcal{B})$ is obtained by traversing each labeled binary tree as in (4.2) according to the following algorithm:

(1) Start from the smallest label (at the root) along the left edge, if possible. If there is no left child, this is the first return to the root (see last step).
(2) At each step, start at the previous vertex and
    (a) try to move away from the root alternating unused left and right edges as far as possible;
    (b) otherwise (if there are no such edges) move towards the root along the same-side edges as far as possible.
(3) The label of the end vertex of this path is the next term in the cycle.
(4) At the first return to the root, the next term is the largest label at the root. At the second return to the root (and when the root has no right child), the cycle is complete.

For example, the traversal of the trees in (4.2) yields the cycles (1573) and (2648).

Note that removal of the nonessential 1s leaves the labels of rows and columns unchanged, and rows still correspond to weak excedances (in particular, rows with no 1s correspond to fixed points), while columns correspond to deficiencies. Thus, to prove Theorem 4.2 we first need to prove the following.

Theorem 4.4. *If $\mathcal{B}$ is a bare tableau with a 1 in every row and a single doubly essential 1, then $\tau = \Theta(B)$ is a cyclic permutation of length greater than 1. Moreover, $\mathcal{B}$ is uniquely recoverable from τ in the cycle notation.*

Proof. The doubly essential 1 (i.e. the root of the corresponding labeled binary tree) must have the labels $(1, n)$ where n is the length of the southeast boundary P of $\mathcal{B}$. Suppose the successive left sons away from the root are labeled $i_1, i_2, \ldots, i_r$. Then $1 < i_1 < i_2 < \cdots < i_r < n$, and each i_{j+1} is the left son of i_j (letting $i_0 := 1$ and $i_{r+1} := n$). Let $D(\mathcal{B}_{i_j})$ be the subtree of $D(\mathcal{B})$ with the right son of i_j as the root. Then, denoting the traversal of $D(\mathcal{B})$ by $\operatorname{tr}(D(\mathcal{B}))$, we have

$$\operatorname{tr}(D(\mathcal{B})) = 1, \operatorname{tr}(D(\mathcal{B}_{i_1})), i_1, \operatorname{tr}(D(\mathcal{B}_{i_2})), i_2, \ldots, \operatorname{tr}(D(\mathcal{B}_{i_r})), i_r, n, \operatorname{tr}(D(\mathcal{B}_n))$$

Each $\mathcal{B}_{i_j}$ is strictly smaller than $\mathcal{B}$, so by inductive assumption the traversal of $\mathcal{B}_{i_j}$ contains every vertex label of $\mathcal{B}_{i_j}$ once. Hence, the traversal of $\mathcal{B}$ contains every label of $\mathcal{B}$ once, i.e. $\tau = \Theta(\mathcal{B})$ is a cyclic permutation.

Note also that the first part of Theorem 2.6 (i.e. that $\pi = \pi' \circ (i_1\, i_2\, \ldots\, i_r\, n)$) remains true for bare tableaux after changing Φ to Θ. Thus, to show that Θ is a bijection we only need to prove that we can uniquely recover the leftmost column of the bare tableau $\mathcal{B}$ with no fixed points and a single doubly essential 1 from the cyclic permutation $\tau = \Theta(\mathcal{B})$.

Now since i_j is a left son, every label in $D(\mathcal{B}_{i_j})$ is larger than i_j. Hence, $(1, i_1, \dots, i_r)$ in reverse order are the successive minima in the direction of the inverse cycle τ^{-1} from n (excluded) to 1 (included), or, in general, from the greatest element to the least. □

Now Theorem 4.4 clearly implies Theorem 4.3, and hence, Theorem 4.2. Note that we can similarly prove that the column labels of vertices in the top row (i.e. row 1) of $\mathcal{B}$ in reverse order are the successive maxima in the direction of the inverse cycle τ^{-1} from 1 (excluded) to n (included), or, in general, from the least element to the greatest, in reverse order.

Example 4.5. If $\tau = (2648)$, then $\mathcal{B} = \theta(\tau)$ has top row label 2 and leftmost column label 8, so the leftmost column of $\mathcal{B}$ has 1s in rows labelled 4 and 2, while the top row of $\mathcal{B}$ only has a 1 in column labeled 8.

Finally, as we remarked in Section 1, this section solves Open Problem 3 of [8], but only partly. Indeed, we have proved that the number of essential 1s of $\mathcal{T}$ is equal to the length minus the number of cycles in some permutation τ related to $\pi = \Phi(\mathcal{T})$ in a somewhat convoluted way as defined earlier in this Section. But part of Open Problem 3 remains unanswered: what do the essential 1s of a tableau $\mathcal{T}$ correspond to in the permutation π itself? To that, we add our accompanying question: what do the doubly essential 1s of $\mathcal{T}$ correspond to in π itself?

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