Restricted Dumont permutations

Alexander Burstein
Department of Mathematics
Iowa State University
Ames, IA 50011-2064 USA

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Abstract

We analyze the structure and enumerate Dumont permutations of the first and second kinds avoiding certain patterns or sets of patterns of length 3 and 4. Some cardinalities are given by Catalan numbers, powers of 2, little Schröder numbers, and other known or related sequences.

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1 Preliminaries

1.1 Patterns

Let $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$ be two permutations. We say that σ *contains* τ, or τ *occurs* in σ, if σ has a subsequence $(\sigma(i_1), \dots, \sigma(i_k))$, $1 \le i_1 < \cdots < i_k \le n$, order-isomorphic to τ. Such a subsequence is called an *occurrence* (or an *instance*) of τ in σ. In this context, the permutation τ is called a *pattern*. We say that σ *avoids* τ, or σ is τ*-avoiding*, if τ does not occur in σ.

Notation 1.1 We denote the set of permutations in $\mathfrak{S}_n$ avoiding a pattern τ by $\mathfrak{S}_n(\tau)$. If T is a set of patterns, then we denote the set of permutations in $\mathfrak{S}_n$ simultaneously avoiding all patterns in T by $\mathfrak{S}_n(T)$.

Permutations avoiding a 3-letter pattern were first considered in [7]. In [10], permutations and involutions avoiding each set T of 3-letter patterns were studied. Since then restricted permutations and forbidden patterns were the subject of many papers. One of the most frequently considered problems is the enumeration of $\mathfrak{S}_n(\tau)$ and $\mathfrak{S}_n(T)$ for various patterns τ and sets of patterns T. The inventory of cardinalities of $|\mathfrak{S}_n(T)|$ for $T \subseteq \mathfrak{S}_3$ is given in [10], and a similar inventory for $|\mathfrak{S}_n(\tau_1, \tau_2)|$, where $\tau_1 \in \mathfrak{S}_3$ and $\tau_2 \in \mathfrak{S}_4$ is given in [15]. Some results on $|\mathfrak{S}_n(\tau_1, \tau_2)|$ for $\tau_1, \tau_2 \in \mathfrak{S}_4$ are obtained in [14]. The exact formula for $|\mathfrak{S}_n(1234)|$ and the generating function for $|\mathfrak{S}_n(12\dots k)|$ are found in [5]. $|\mathfrak{S}_n(1342)| = |\mathfrak{S}_n(1423)|$ is obtained in [2], and [12, 13] shows that $|\mathfrak{S}_n(3142)| = |\mathfrak{S}_n(1342)|$. For a survey of results on pattern avoidance, see [1, 6].

Example 1.2

- $|\mathfrak{S}_n(\tau)| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number, for any $\tau \in \mathfrak{S}_3$.
- $|\mathfrak{S}_n(123, 213)| = |\mathfrak{S}_n(132, 231)| = 2^{n-1}$.
- $|\mathfrak{S}_n(123, 132, 213)| = F_n$, the nth Fibonacci number.
- $|\mathfrak{S}_n(3142, 2413)| = |\mathfrak{S}_n(4132, 4231)| = |\mathfrak{S}_n(2431, 4231)| = r_{n-1}$, the nth large Schröder number [11, Sequence A006318], given by $r_0 = 1$, $r_n = r_{n-1} + \sum_{j=0}^{n-1} r_k r_{n-k}$.

Another problem is finding (sets of) patterns T_1 and T_2 such that $|\mathfrak{S}_n(T_1)| = |\mathfrak{S}_n(T_2)|$ for any $n \geq 0$. Such (sets of) patterns are called *Wilf-equivalent* and said to belong to the same *Wilf class*. There are three symmetry operations on $\mathfrak{S}_k$ that map every pattern onto a Wilf-equivalent pattern:

- *reversal* r: $r(\tau)(i) = \tau(n+1-i)$, i.e. $r(\tau)$ is τ read right-to-left.
- *complement* c: $c(\tau)(i) = n+1-\tau(i)$, i.e. $c(\tau)$ is τ read upside down.
- $r \circ c = c \circ r$: $r \circ c(\tau)(i) = n+1-\tau(n+1-i)$, i.e. $r \circ c(\tau)$ is τ read right-to-left upside down.

The set of patterns $\{\tau, r(\tau), c(\tau), r(c(\tau))\}$ is called the *symmetry class* of τ.

1.2 Dumont permutations

Definition 1.3 A *Dumont permutation of the first kind* is a permutation $\pi \in \mathfrak{S}_{2n}$ where each even entry is followed by a descent and each odd entry is followed by an ascent or ends the string. In other words, for every $i = 1, 2, \ldots, 2n$,

$$\pi(i) \text{ is even} \implies i < 2n \text{ and } \pi(i) > \pi(i+1),$$
$$\pi(i) \text{ is odd} \implies \pi(i) < \pi(i+1) \text{ or } i = 2n.$$

Definition 1.4 A *Dumont permutation of the second kind* is a permutation $\pi \in \mathfrak{S}_{2n}$ where all entries at even positions are deficiencies and all entries at odd positions are fixed points or excedances. In other words, for every $i = 1, 2, \ldots, n$,

$$\pi(2i) < 2i,$$
$$\pi(2i+1) \geq 2i+1.$$

Notation 1.5 We denote the set of Dumont permutations of the first (resp. second) kind of length $2n$ by $\mathfrak{D}^1_{2n}$ (resp. $\mathfrak{D}^2_{2n}$).

Example 1.6 $\mathfrak{D}^1_2 = \mathfrak{D}^2_2 = \{21\}$, $\mathfrak{D}^1_4 = \{2143, 3421, 4213\}$, $\mathfrak{D}^2_4 = \{2143, 3142, 4132\}$.

Remark 1.7 Dumont permutations of odd length can be defined similarly to those of even length. Then $\mathfrak{D}^1_{2n+1}$ or $\mathfrak{D}^2_{2n+1}$ are obtained simply by adjoining $2n+1$ to the end of each permutation in $\mathfrak{D}^1_{2n}$ or $\mathfrak{D}^2_{2n}$, respectively. Obviously, $|\mathfrak{D}^1_{2n+1}| = |\mathfrak{D}^1_{2n}|$ and $|\mathfrak{D}^2_{2n+1}| = |\mathfrak{D}^2_{2n}|$.

Dumont [4] showed that

$$|\mathfrak{D}^1_{2n}| = |\mathfrak{D}^2_{2n}| = G_{2n+2} = 2(1-2^{2n+2})B_{2n+2},$$

where G_n is the nth Genocchi number, a multiple of the Bernoulli number B_n. Lists of Dumont permutations $\mathfrak{D}^1_{2n}$ and $\mathfrak{D}^2_{2n}$ for $n \le 4$ as well as some basic information and references for Genocchi numbers and Dumont permutations may be obtained at [11, A001469] and [9]. We only note that the exponential generating functions for the unsigned and signed Genocchi numbers are given by

$$\sum_{n=1}^{\infty} G_{2n}\frac{x^{2n}}{(2n)!} = x\tan\frac{x}{2}, \qquad \sum_{n=1}^{\infty} (-1)^n G_{2n}\frac{x^{2n}}{(2n)!} = \frac{2x}{e^x+1} - x = -x\tanh\frac{x}{2}.$$

Throughout this paper we will use the following obvious facts about Dumont permutations:

- in any $\pi \in \mathfrak{D}^1_{2n}$, 2 is always followed by 1, and $2n-1$ is followed by $2n$ or is final in π;
- in any $\pi \in \mathfrak{D}^2_{2n}$, $\pi(2) = 1$ and $\pi(2n-1) = 2n-1$ or $2n$.

We define Wilf-equivalence and Wilf classes on Dumont permutations in the same way as Wilf-equivalence on all permutations (and call it $\mathfrak{D}^1$-Wilf-equivalence or $\mathfrak{D}^2$-Wilf-equivalence according to the kind). Note that since reversal, complement, or reversal of complement of Dumont permutations are no longer Dumont permutations, it follows that permutations in the same symmetry class are not necessarily Wilf-equivalent on Dumont permutations.

Remark 1.8 Sometimes slightly different permutations are defined as Dumont permutations of either kind. Those will be useful later on, and we will describe them now.

Permutations $\pi \in \mathfrak{S}_{2n}$ in which each odd entry is followed by an ascent and each even entry is followed by a descent or ends the string are obtained by applying the complement operation c to our Dumont permutations of the first kind, and we will call them *Dumont-like permutations of the first kind* and denote the set of these in $\mathfrak{S}_n$ by $\mathcal{D}^1_n$.

Similarly, permutations $\pi \in \mathfrak{S}_{2n}$ with $\pi(2i+1) > 2i+1$ and $\pi(2i) \le 2i$ for all i are obtained by applying the operation $r \circ c = c \circ r$ to our Dumont permutations of the second kind, and we will call them *Dumont-like permutations of the second kind* and denote the set of these in $\mathfrak{S}_n$ by $\mathcal{D}^2_n$.

1.3 Restricted Dumont permutations

So far, there has been a single paper on restricted Dumont permutations, namely Mansour [8]. Most of it is devoted to the study of 132-avoiding Dumont permutations of the first kind (as there are no 132-avoiding Dumont permutations of the second kind other than $21 \in \mathfrak{D}^2_2$). Specifically, 132-avoiding Dumont permutations of the first kind which also avoid (contain exactly once) certain other patterns τ are examined and enumerated. However, Dumont permutations avoiding other patterns are briefly examined as well.

Notation 1.9 We denote the set of permutations in $\mathfrak{D}_n^1$ or $\mathfrak{D}_n^2$ avoiding a pattern τ by $\mathfrak{D}_n^1(\tau)$ or $\mathfrak{D}_n^2(\tau)$, respectively. If T is a set of patterns, then we denote the set of permutations in $\mathfrak{D}_n^1$ or $\mathfrak{D}_n^2$ avoiding all patterns in T by $\mathfrak{D}_n^1(T)$ or $\mathfrak{D}_n^2(T)$, respectively. We define $\mathcal{D}_n^1(T)$ and $\mathcal{D}_n^2(T)$ (see Remark 1.8) similarly.

Remark 1.10 Note that $\mathcal{D}_{2n}^1(T) = c(\mathfrak{D}_{2n}^1(c(T)))$ and $\mathcal{D}_{2n}^2(T) = r \circ c(\mathfrak{D}_{2n}^1(r \circ c(T)))$, so $|\mathcal{D}_{2n}^1(T)| = |\mathfrak{D}_{2n}^1(c(T))|$ and $|\mathcal{D}_{2n}^2(T)| = |\mathfrak{D}_{2n}^2(r \circ c(T))|$.

Theorem 1.11 ([8, Theorems 2.2, 4.3]) $|\mathfrak{D}_{2n}^1(132)| = |\mathfrak{D}_{2n}^1(231)| = |\mathfrak{D}_{2n}^1(312)| = |\mathfrak{D}_{2n}^2(321)| = C_n$. *Similarly,* $|\mathfrak{D}_{2n+1}^1(132)| = |\mathfrak{D}_{2n+1}^1(231)| = |\mathfrak{D}_{2n+1}^1(312)| = |\mathfrak{D}_{2n+1}^2(321)| = C_n$.

Another bit of notation will be useful before we proceed.

Notation 1.12 Let π' and π'' be subsequences of a permutation π. We say that $\pi' > \pi''$ if every entry of π' is greater than every entry of π''. Also, for a permutation π and an integer m, the string $\pi + m = m + \pi$ is obtained by adding m to every entry of π. We define $m - \pi$ similarly. For the empty string, we define $m \pm \emptyset = \emptyset$ for any m.

All cardinalities may be obtained similarly by examining the recursive structure of restricted permutations. For example, it immediately follows from [8, Proposition 2.1] that a permutation in $\mathfrak{D}_{2n}^1(132)$ can be of two types:

1. $\pi = (\pi', 2n-1, 2n, \pi'')$, where $\pi' > \pi''$ and $\pi'' \in \mathfrak{D}_{2k}^1(132)$ and $\pi' - 2k \in \mathfrak{D}_{2n-2k-2}^1(132)$ for some $1 \le k \le n-1$.

2. $\pi = (2n, \pi', 2n-1)$, where $\pi' \in \mathfrak{D}_{2n-2}^1(132)$.

(All permutations are given as lists in one-line notation unless otherwise indicated.)

The fact that $\pi' > \pi''$ follows from the fact that π avoids 132. To see that $\pi'' \in \mathfrak{D}_{2k}^1$ (and not $\mathfrak{D}_{2k+1}^1$) in the first case, note that the minimum element of π' must be odd since it must be followed by an ascent.

Other patterns mentioned in Theorem 1.11 are treated similarly.

In this paper, we use the same approach to analyze and enumerate Dumont permutations of either kind avoiding certain patterns or pairs of patterns of length 3 or 4. We will show that symmetry operations on Dumont permutations do not necessarily produce a set of patterns in the same Wilf class, hence we will only consider certain cases in detail. We will see that the some cardinalities of sets avoiding a given set of patterns are given by Catalan numbers, powers of 2, little Schröder numbers $s_n = r_n/2$, and other known sequences.

We will frequently make use of Remark 1.10, especially when considering $\mathfrak{D}_{2n}^1(T)$ for a set of patterns $T = c(T)$, such as $\{3142, 2413\}$, $\{1342, 4213\}$ or $\{1423, 4132\}$, and $\mathfrak{D}_{2n}^2(T)$ for a pattern or a set of patterns $T = r \circ c(T)$, such as $T = \{3142\}$.

2 Dumont permutations avoiding 3-letter patterns

As we mentioned before, Theorem 1.11 gives $|\mathfrak{D}_{2n}^1(132)| = |\mathfrak{D}_{2n}^1(231)| = |\mathfrak{D}_{2n}^1(312)| = |\mathfrak{D}_{2n}^2(321)| = C_n$.

Theorem 2.1 $|\mathfrak{D}^1_{2n}(213)| = C_{n-1}$ *for* $n \geq 1$.

Note that $213 = c(231)$ but $|\mathfrak{D}^1_{2n}(213)| \neq |\mathfrak{D}^1_{2n}(231)|$ whereas $132 = c(312)$ and $|\mathfrak{D}^1_{2n}(132)| = |\mathfrak{D}^1_{2n}(312)|$. Thus, patterns in the same symmetry class are not necessarily $\mathfrak{D}^1$-Wilf-equivalent (or $\mathfrak{D}^2$-Wilf-equivalent as we will see below).

Proof. Since 1 immediately follows 2 in any permutation $\pi \in \mathfrak{D}^1_{2n}(213)$, we see that π must end on 21. Clearly, $\mathfrak{D}^1_2(213) = \{21\}$ (and hence $|\mathfrak{D}^1_2(213)| = 1 = C_0$), so consider $n \geq 2$. Let $\pi(1) = j$. Then $\pi = (j, \pi_1, \pi_2)$ for some permutations $\pi_1 > j > \pi_2$. Then $\pi_2 = (\pi'', 2, 1)$, so $j \geq 3$. Since j is the minimal entry of (j, π_1), it follows that j must be odd, i.e. $j = 2k+1$ for some $k \geq 1$. Since $(2k+1, \pi_1) > \pi_2$ and $(2k+1, \pi_1)$ starts with $2k+1$, it follows that $(2k+1, \pi_1)$ ends with $2k+2$. Let $(2k+1, \pi_1) = (2k+1, \pi', 2k+2)$, then the last letter of π' is even, so $\pi' - (2k+2) \in \mathcal{D}^1_{2n-2k-2}(213)$, in other words, $c(\pi' - (2k+2)) = 2n+1-\pi' \in \mathfrak{D}^1_{2n-2k-2}(231)$. Similarly, $\pi_2 = (\pi'', 2, 1)$, so the last letter of π'' is even and hence $\pi'' - 2 \in \mathcal{D}^1_{2k-2}(213)$, i.e. $c(\pi'' - 2) = 2k+1-\pi'' \in \mathfrak{D}^1_{2k-2}(231)$.

Thus, the set $\mathfrak{D}^1_{2n}(213)$ for $n \geq 1$ consists of all permutations

$$\pi = (2k+1, c(\rho') + 2k+2, 2k+2, c(\rho'') + 2, 2, 1)$$

for some $k = 1, 2, \ldots, n-1$, and some $\rho' \in \mathfrak{D}^1_{2n-2k-2}(231)$ and $\rho'' \in \mathfrak{D}^1_{2k-2}(231)$. Therefore, Theorem 1.11 implies that, for $n \geq 2$,

$$|\mathfrak{D}^1_{2n}(213)| = \sum_{k=1}^{n-1} C_{n-1-k} C_{k-1} = C_{n-1}.$$

Note that we could have done without the result of Theorem 1.11 and proved our result using the recurrence relation alone, but using Theorem 1.11 makes for a slight shortcut in our argument. □

Theorem 2.2 $|\mathfrak{D}^2_{2n}(231)| = 2^{n-1}$ *for* $n \geq 1$.

Proof. Since π avoids 231, if follows that for any entry j of π, every element $< j$ to the left of j must be lesser than every element $< j$ to the right of j. If $\pi \in \mathfrak{D}^2_{2n}(231)$, then $\pi(2n-1) = 2n$ or $\pi(2n-1) = 2n-1$, otherwise $(2n-1, 2n, \pi(2n))$ is an instance of pattern 231 in π. We also have $\pi(2n) < 2n$. Therefore, $\pi(2n-1) = 2n$ implies $\pi(2n) = 2n-1$, and $\pi(2n-1) = 2n-1$ implies $\pi(2n) = 2n-2$. In these two cases, the last two entries of π cannot be part of any occurrence of 231. Thus, $\pi \in \mathfrak{D}^2_{2n}(231)$ if and only if either of the following two cases holds:

- $\pi = (\pi', 2n, 2n-1)$ for any $\pi' \in \mathfrak{D}^2_{2n-2}(231)$
- $\pi = (\widehat{\pi}', 2n-1, 2n-2)$ for any $\pi' \in \mathfrak{D}^2_{2n-2}(231)$, where $\widehat{\pi}'$ is obtained by replacing $2n-2$ with $2n$ in π'.

Therefore, $|\mathfrak{D}^2_{2n}(231)| = 2 \cdot |\mathfrak{D}^2_{2n-2}(231)|$ for $n \geq 2$, and $|\mathfrak{D}^2_2(231)| = 1$, so $|\mathfrak{D}^2_{2n}(231)| = 2^{n-1}$ for $n \geq 1$. □

This recursive description allows us to determine the cycle structure of permutations in $\mathfrak{D}^2_{2n}(231)$. It is easy to see inductively that any $\pi \in \mathfrak{D}^2_{2n}(231)$ has n cycles, each of which contains exactly one odd entry and is of the form $(2k-1)$ or $(2l, 2l-2, \ldots, 2k, 2k-1)$ for some $0 \le k \le l \le n$. Clearly, the above cycle must be followed by $l-k$ fixed points $(2i-1)$, $k+1 \le i \le l$. Thus, there is a natural bijection between permutations $\mathfrak{D}^2_{2n}(231)$ with $n-k$ fixed points and weak k-compositions of n (of which there are $\binom{n-1}{k-1}$), where each cycle $(2l, 2l-2, \ldots, 2k, 2k-1)$ with $k \le l$ is mapped onto a part of size $l-k+1$ (and fixed points are "forgotten").

$$\mathfrak{D}^2_8(231) \ni 21835476 = (21)(8643)(5)(7) \mapsto 4 = 1+3$$
$$4 = 1+3 \mapsto \underbrace{1}_{1} + \underbrace{3+0+0}_{3} \mapsto (21)(8643)(5)(7) \in \mathfrak{D}^2_8(231).$$

There is at most one Dumont permutation avoiding a given pair of 3-letter patterns simultaneously (except $|\mathfrak{D}^1_{2n}(123, 132)| = 2$). The following results are proved in the same way as Theorems 2.1 and 2.2, and we list them here for completeness.

$$\begin{aligned}
\mathfrak{D}^1_{2n}(321) = \mathfrak{D}^2_{2n}(312) &= \{(2, 1, 4, 3, \ldots, 2n, 2n-1)\} \\
\mathfrak{D}^1_{2n}(132, 231) &= \{(2n, 2n-2, \ldots, 4, 2, 1, 3, \ldots, 2n-3, 2n-1)\} \\
\mathfrak{D}^1_{2n}(132, 312) &= \{(\ldots, 2n-3, 2n-2, 4, 3, 2n-1, 2n, 2, 1)\} \\
\mathfrak{D}^1_{2n}(213, 312) &= \{(3, 5, \ldots, 2n-1, 2n, \ldots, 6, 4, 2, 1)\} \\
\mathfrak{D}^1_{2n}(123, 213) = \mathfrak{D}^1_{2n}(132, 213) &= \{(2n-1, 2n, \ldots, 5, 6, 3, 4, 2, 1)\} \\
\mathfrak{D}^2_{2n}(231, 321) = \mathfrak{D}^1_{2n}(231, 312) &= \{(2, 1, 4, 3, \ldots, 2n, 2n-1)\}
\end{aligned}$$

Similarly, we have $\mathfrak{D}^1_{2n}(123) = \{(2n-1, 2n, 2n-3, 2n-2, \ldots, 7, 8, \pi) \mid \pi \in \mathfrak{D}^1_6(123) = \{436215, 562143, 563421, 564213\}\}$, $\mathfrak{D}^2_{2n}(123) = \mathfrak{D}^2_{2n}(213) = \emptyset$ for $n \ge 3$, and $\mathfrak{D}^2_{2n}(132) = \mathfrak{D}^1_{2n}(213, 231) = \emptyset$ for $n \ge 2$.

3 Dumont permutations avoiding 4-letter patterns

When considering sets $\mathfrak{S}_n(T)$ of permutations avoiding a given set of patterns T, the first nontrivial cases arise when T is a single 3-letter permutation [7, 10]. In our case, the first nontrivial restrictions by Dumont permutations are by singletons in $\mathfrak{D}^1_4 = \{2143, 3421, 4213\}$ and $\mathfrak{D}^2_4 = \{2143, 3142, 4132\}$. Here we consider one of these six cases, that of $3142 \in \mathfrak{D}^2_4$. The patterns $2143, 4132 \in \mathfrak{D}^2_4$ are treated in [3].

3.1 Avoiding a single pattern

Theorem 3.1 *$|\mathfrak{D}^2_{2n}(3142)| = C_n$ for $n \ge 0$.*

Proof. Let $\pi \in \mathfrak{D}^2_{2n}(3142)$. Then $\pi(2) = 1$. Consider two cases based on parity of $\pi(1)$. Suppose that $\pi(1) = 2k-1$ for some $1 \le k \le n$, then $\pi(2k-1) > 2k-1$. If $\pi(i) < 2k-1$ for some $i > 2k-1$, then $(\pi(1), \pi(2), \pi(2k-1), \pi(i))$ is an occurrence of pattern 3142 in π. Therefore, $2 \le \pi(i) \le 2k-2$ only if $3 \le i \le 2k-2$, i.e. all entries in $\{2, 3, 4 \ldots, 2k-2\}$ must occupy positions in $\{3, 4, \ldots, 2k-2\}$, which is impossible.

Hence, we must have $\pi(1) = 2k$ for some $1 \le k \le n$. Since $\pi(2k+1) > 2k$, it follows, as before, that $2 \le \pi(i) \le 2k-1$ only if $3 \le i \le 2k$, i.e. all entries in $\{2, 3, \ldots, 2k-1\}$

must occupy positions in $\{3, 4, \ldots, 2k\}$. In other words, $\pi = (2k, 1, \pi_1 + 1, \pi_2 + 2k)$, where π_1 is a certain permutation of $[2k-2] = \{1, 2, \ldots, 2k-2\}$ avoiding pattern 3142, and $\pi_2 \in \mathfrak{D}^2_{2n-2k}(3142)$. Furthermore, since $\pi_1 + 1$ is a segment of $\pi \in \mathfrak{D}^2_{2n}(3142)$ starting at position 3, it is easy to see that $\pi_1 \in \mathcal{D}^2_{2k-2}(3142)$, so $\pi' = (r \circ c)(\pi_1) \in \mathfrak{D}^2_{2k-2}(3142)$ since $(r \circ c)(3142) = 3142$. Thus, $\pi \in \mathfrak{D}^2_{2n}(3142)$ if and only if $\pi = (2k, 1, (r \circ c)(\pi') + 1, \pi'' + 2k)$ for some $k = 1, 2, \ldots, n$, and any $\pi' \in \mathfrak{D}^2_{2k-2}(3142)$ and $\pi'' \in \mathfrak{D}^2_{2n-2k}(3142)$. If we let $a_n = |\mathfrak{D}^2_{2n}(3142)|$, then $a_0 = 1$ and $a_n = \sum_{k=1}^{n} a_{k-1}a_{n-k}$ for $n \geq 1$, so $a_n = C_n$. □

In fact, from the above proof, it is easy to see that any $\pi \in \mathfrak{D}^2_{2n}(3142)$ has the form

$$\pi = (\pi_1, \pi_2 + |\pi_1|, \pi_3 + |\pi_2| + |\pi_1|, \ldots, \pi_m + |\pi_{m-1}| + \cdots + |\pi_1|)$$

for some weak m-composition $n = n_1 + n_2 + \cdots + n_m$ (i.e. all $n_i > 0$), and $\pi_i \in \mathfrak{D}^2_{2n_i}(3142)$ ($i = 1, 2, \ldots, m$) is of the form

$$\pi_i = (2n_i, 1, (r \circ c)(\pi_i')), \quad \pi_i' \in \mathfrak{D}^2_{2n_i-2}(3142).$$

Furthermore, $(2n_1, 2(n_1+n_2), 2(n_1+n_2+n_3), \ldots, 2n)$ is the sequence of left-to-right maxima of π. A natural bijective map from $\mathfrak{D}_{2n}(3142)$ to Dyck paths of $2n$ steps can be derived recursively from the above decomposition, where left-to-right maxima of π (see above) are mapped to the steps leaving the x-axis, and entries immediately following them are mapped to the steps returning to the x-axis.

Since 321 is a subpattern of 4132, we must have $\mathfrak{D}^2_{2n}(321) \subseteq \mathfrak{D}^2_{2n}(4132)$. It is easy to see that $\mathfrak{D}^2_{2n}(321)$ consists of all permutations in $\mathfrak{D}^2_{2n}$ in which both the subsequence of entries in even positions and the subsequence of entries in odd positions are increasing. It has been proved in [3] that $\mathfrak{D}^2_{2n}(4132)$ consists of these permutations only:

Theorem 3.2 ([3]) $\mathfrak{D}^2_{2n}(4132) = \mathfrak{D}^2_{2n}(321)$ *and* $|\mathfrak{D}^2_{2n}(4132)| = |\mathfrak{D}^2_{2n}(321)| = C_n$ *for* $n \geq 0$.

So far, we were unable to determine other cardinalities with single pattern restrictions. The following lemma gives a relationship between two such patterns, 4213 and 1342 = $c(4213)$.

Lemma 3.3 *If* $A(x)$ *and* $B(x)$ *are ordinary generating functions for* $|\mathfrak{D}^1_{2n}(4213)|$ *and* $|\mathfrak{D}^1_{2n}(1342)|$, *respectively, then*

$$A(x) = \frac{1}{1 - xB(x)}.$$

Proof. Let $\pi \in \mathfrak{D}^1_{2n}(4213)$. If $n > 0$, then $\pi = (\pi_1, 2, 1, \pi_2)$ for some $\pi_1 < \pi_2$. Moreover, it easy to see that $|\pi_1| + |\pi_2| = 2n - 2$, $\pi_1 - 2 \in \mathcal{D}^1(4213)$ so $\pi' = |\pi_1| + 3 - \pi_1 \in \mathfrak{D}^1(1342)$, and $\pi'' = \pi_2 - |\pi_1| \in \mathfrak{D}^1(4213)$. It follows that

$$A(x) = 1 + B(x)xA(x),$$

which implies the lemma. □

Among single 4-letter Dumont permutations of either kind, only 3142 and 4132 are $\mathfrak{D}^2$-Wilf-equivalent. It is easy to see that 2143 is not $\mathfrak{D}^2$-Wilf-equivalent to 3142, and as the following table shows, no two of the patterns in $\mathfrak{D}^1_4$ are $\mathfrak{D}^1$-Wilf-equivalent.

n	0	1	2	3	4	5
$\lvert\mathfrak{D}^1_{2n}(3421)\rvert$	1	1	2	7	36	241
$\lvert\mathfrak{D}^1_{2n}(2143)\rvert$	1	1	2	7	36	239
$\lvert\mathfrak{D}^1_{2n}(4213)\rvert$	1	1	2	6	25	135

3.2 Avoiding a pair of patterns

Theorem 3.4 $|\mathfrak{D}^1_{2n}(1342, 1423)| = s_{n+1}$ *for* $n \geq 0$.

Here s_n is the nth little Schröder number [11, A001003], given by $s_{n+1} = -s_n + 2\sum_{k=1}^{n} s_k s_{n-k}$ ($n \geq 2$), $s_1 = 1$, and the generating function $s(x) = \sum_{n=1}^{\infty} s_n x^n = \left(1 + x - \sqrt{1 - 6x + x^2}\right)/4$.

Proof. If $\pi \in \mathfrak{D}^1_{2n}(1342, 1423)$, then π can be of two types:

- $\pi = (\pi_1, 2n-1, 2n, \pi_2)$ with $\pi_1 = \emptyset$ or $\pi_1 > \pi_2 \neq \emptyset$ (otherwise π contains 1342); or,
- $\pi = (\pi_1, 2n, \pi_2, 2n-1)$ with $\pi_1 > \pi_2$ or $\pi_1 = \emptyset$ or $\pi_2 = \emptyset$ (otherwise π contains 1423).

In either case, if $\pi_1 \neq \emptyset$, then the minimum entry of π_1 is odd. Hence, the maximum entry of π_2 is even, whether or not π_1 is nonempty. For π of either type, the entries $2n-1$ and $2n$ cannot be part of any occurrence of 1342 or 1423, so π avoids 1342 and 1423 if and only if both π_1 and π_2 avoid 1342 and 1423. Thus, $\pi \in \mathfrak{D}^1_{2n}(1342, 1423)$ if and only if

- $\pi = (\pi' + 2k, 2n-1, 2n, \pi'')$ for $1 \leq k \leq n-1$, $\pi' \in \mathfrak{D}^1_{2n-2k-2}(1342, 1423)$, $\pi'' \in \mathfrak{D}^1_{2k}(1342, 1423)$; or,
- $\pi = (\pi' + 2k, 2n, \pi'', 2n-1)$ for $0 \leq k \leq n-1$, $\pi' \in \mathfrak{D}^1_{2n-2k-2}(1342, 1423)$, $\pi'' \in \mathfrak{D}^1_{2k}(1342, 1423)$.

Therefore, for $a_n = |\mathfrak{D}^1_{2n}(1342, 1423)|$, we have $a_0 = 1$ and $a_n = -a_{n-1} + 2\sum_{k=0}^{n-1} a_k a_{n-1-k}$, so $\{a_n\}_{n\geq 0}$ satisfies the same recurrence relation as $\{s_{n+1}\}_{n\geq 0}$, and hence, $a_n = s_{n+1}$. □

Theorem 3.5 $|\mathfrak{D}^1_{2n}(2341, 2413)| = s_{n+1}$ *for* $n \geq 0$.

Proof. The proof is very similar to that of Theorem 3.4. We have $\pi \in \mathfrak{D}^1_{2n}(2341, 2413)$ if and only if

- $\pi = (\pi', 2n-1, 2n, \pi'' + 2k)$ for $0 \leq k \leq n-2$, $\pi' \in \mathfrak{D}^1_{2k}(2341, 2413)$, $\pi'' \in \mathfrak{D}^1_{2n-2k-2}(2341, 2413)$; or,
- $\pi = (\pi', 2n, \pi'' + 2k, 2n-1)$ for $0 \leq k \leq n-1$, $\pi' \in \mathfrak{D}^1_{2k}(2341, 2413)$, $\pi'' \in \mathfrak{D}^1_{2n-2k-2}(2341, 2413)$,

Hence, the same recursive relation is obtained, and the theorem follows. □

Theorem 3.6 $|\mathfrak{D}^1_{2n}(1342, 2413)| = s_{n+1}$ *for* $n \geq 0$.

Proof. Again, the proof is very similar to that of Theorem 3.4. We have $\pi \in \mathfrak{D}^1_{2n}(1342, 2413)$ if and only if

- $\pi = (\pi' + 2k, 2n-1, 2n, \pi'')$ for $1 \leq k \leq n-1$, where $\pi' \in \mathfrak{D}^1_{2n-2k-2}(1342, 2413)$, $\pi'' \in \mathfrak{D}^1_{2k}(1342, 2413)$; or,

- $\pi = (\pi', 2n, \pi'' + 2k, 2n-1)$ for $0 \le k \le n-1$, where $\pi' \in \mathfrak{D}^1_{2k}(1342, 2413)$, $\pi'' \in \mathfrak{D}^1_{2n-2k-2}(1342, 2413)$.

Hence, the same recursive relation is obtained, and the theorem follows. □

Theorem 3.7 $|\mathfrak{D}^1_{2n}(2341, 1423)| = a_n$*, where* a_n *satisfies the recurrence relation* $a_n = 3a_{n-1} + 2a_{n-2}$ *for* $n \ge 3$*, with* $a_0 = 1$*,* $a_1 = 1$*,* $a_2 = 3$*.*

Note that this sequence is [11, A007482], shifted right by one position. In other words, $a_n = |\mathfrak{D}^1_{2n}(2341, 1423)|$ is the number of subsets of $[2n-2] = \{1, 2, \ldots, 2n-2\}$ where each odd element m has an even neighbor ($m-1$ or $m+1$). We also have $a_0 = 1$, $a_1 = 1$, $a_2 = 3$, and $a_n = 3a_{n-1} + 2a_{n-2}$ for $n \ge 3$.

Proof. As before, we have $\pi \in \mathfrak{D}^1_{2n}(2341, 1423)$ only if

1. $\pi = (\pi', 2n-1, 2n, \pi'' + 2k)$ for $0 \le k \le n-2$, $\pi' \in \mathfrak{D}^1_{2k}(2341, 1423)$, $\pi'' \in \mathfrak{D}^1_{2n-2k-2}(2341, 1423)$; or,
2. $\pi = (\pi' + 2k, 2n, \pi'', 2n-1)$ for $0 \le k \le n-1$, $\pi' \in \mathfrak{D}^1_{2n-2k-2}(2341, 1423)$, $\pi'' \in \mathfrak{D}^1_{2k}(2341, 1423)$.

However, now the entries $2n-1$ and $2n$ may be part of an occurrence of a pattern 1423 (in case 1) or a pattern 2341 (in case 2). In fact, it is easy to see that for $n \ge 3$ we must have $k \in \{0, n-2\}$ in case 1, and $k \in \{0, n-2, n-1\}$ in case 2. In other words, $\pi \in \mathfrak{D}^1_{2n}(2341, 1423)$ if and only if one of the following holds:

1. $\pi = (2n-1, 2n, \pi')$ for any $\pi' \in \mathfrak{D}^1_{2n-2}(2341, 1423)$.
2. $\pi = (\pi', 2n-1, 2n, 2n-2, 2n-3)$ for any $\pi' \in \mathfrak{D}^1_{2n-4}(2341, 1423)$.
3. $\pi = (2n, \pi', 2n-1)$ for any $\pi' \in \mathfrak{D}^1_{2n-2}(2341, 1423)$.
4. $\pi = (\pi', 2n, 2n-1)$ for any $\pi' \in \mathfrak{D}^1_{2n-2}(2341, 1423)$.
5. $\pi = (2n-2, 2n-3, 2n, \pi', 2n-1)$ for any $\pi' \in \mathfrak{D}^1_{2n-4}(2341, 1423)$.

Therefore, $a_n = 3a_{n-1} + 2a_{n-2}$ for $n \ge 3$, and the theorem follows. □

Theorem 3.8 $\mathfrak{D}^1_{2n}(231, 4213) = \{(2, 1, 4, 3, \ldots, 2n, 2n-1)\}$ *for* $n \ge 1$*.*

Proof. For $n = 1$, the result is obvious. Let $n \ge 2$ and $\pi \in \mathfrak{D}^1_{2n}(231, 4213)$. If $\pi(2n) \ne 2n-1$, then π contains a segment $(2n-1, 2n, j)$ for some $j < 2n-1$, which is an instance of pattern 231. Hence, $\pi(2n) = 2n-1$. Now suppose that $\pi(2n-1) \ne 2n$. If $2n-2$ occurs between $2n$ and $2n-1$, then $2n-2$ must be followed by some $j < 2n-2$, so $(2n, 2n-2, j, 2n-1)$ is an instance of pattern 4213. Therefore, $2n-2$ occurs before $2n$, so $2n$ must be followed by some $j < 2n-2$, and hence $(2n-2, 2n, j)$ is an instance of pattern 231. Thus, $\pi(2n-1) = 2n$, in other words, $\pi = (\pi', 2n, 2n-1)$ for some $\pi' \in \mathfrak{D}^1_{2n-2}(231, 4213)$. The rest is obvious. □

Theorem 3.9 $|\mathfrak{D}^1_{2n}(1342, 4213)| = 2^{n-1}$ *for* $n \ge 1$*.*

Proof. Let $\pi \in \mathfrak{D}^1_{2n}(1342, 4213)$, then $\pi = (\pi_1, 2, 1, \pi_2)$, so $\pi_1 < \pi_2$ if both $\pi_1, \pi_2 \neq \emptyset$. Hence, the largest letter in π_1 is even (since it must be followed by a descent) and the last letter of π_1 is even (since it is followed by 2). Also, π_2 must avoid 231 since $(2, 1, \pi_2)$ avoids 1342. Therefore, $\pi = (\pi' + 2, 2, 1, \pi'' + 2k)$ for some $\pi' \in \mathcal{D}^1_{2k-2}(1342, 4213)$ (which means $c(\pi') \in \mathfrak{D}^1_{2k-2}(1342, 4213)$) and $\pi'' \in \mathfrak{D}^1_{2n-2k}(231, 4213)$. Hence, either $\pi'' = \emptyset$, in which case $\pi = (2n + 1 - c(\bar{\pi}), 2, 1)$ for some $\bar{\pi} \in \mathfrak{D}^1_{2n-2}(1342, 4213)$, or $\pi'' \neq \emptyset$, in which case $\pi = (\bar{\pi}, 2n, 2n - 1)$ for some $\bar{\pi} \in \mathfrak{D}^1_{2n-2}(1342, 4213)$. The rest is obvious. □

Notation 3.10 For the remaining part we will define $C(x) = \frac{1-\sqrt{1-4x}}{2x}$, the ordinary generating function for the sequence of Catalan numbers.

Theorem 3.11 *The ordinary generating function for* $|\mathfrak{D}^1_{2n}(2413, 3142)|$ $(n \geq 0)$ *is given by*

$$F(x) = \frac{3 - \sqrt{1 - 8x}}{2(1 + x)} = \frac{1}{3} C\left(\frac{2(1 + x)}{9}\right) = \frac{1 + 2xC(2x)}{1 + x} = \frac{1}{1 - xC(2x)},$$

so

$$|\mathfrak{D}^1_{2n}(2413, 3142)| = (-1)^n + \sum_{k=1}^{n} (-1)^{n-k} 2^k C_{k-1} = \sum_{k=0}^{n-1} \frac{n - k}{n + k} \binom{n + k}{n} 2^k + \delta_{n=0}$$

is the convolution of ballot numbers and powers of 2.

Note also that $3142 = c(2413)$, and that $|\mathfrak{D}^1_{2n}(2413, 3142)| = C(2; n)$, the generalized Catalan number [11, A064062].

Proof. Let $\pi \in \mathfrak{D}^1_{2n}(2413, 3142)$ $(n \geq 1)$, and suppose that $\pi(2n) = d = 2k - 1$ for some $1 \leq k \leq n$. Consider a subsequence (a, b, c, d) of π. If $a < d$, $b > d$ and $c < d$, then $a < c$ since π avoids 2413. Similarly, if $a > d$, $b < d$ and $c > d$, then $a > c$ since π avoids 3142. Therefore,

$$\pi = (\ldots, \pi_6, \pi_5, \pi_4, \pi_3, \pi_2, 2k - 2, \pi_1, 2k - 1),$$

where $2k - 1 > \pi_1 > \pi_3 > \pi_5 > \ldots$ and $2k - 2 < \pi_2 < \pi_4 < \pi_6 < \ldots$. Also, π_1 and π_2 may be empty, and while each π_i for $i \geq 3$ must be nonempty, the sequence $(\ldots, \pi_6, \pi_5, \pi_4, \pi_3)$ may be empty. Note that, if i is odd (resp. even), then each nonempty π_i is followed by ascent (resp. descent), hence must end on an odd (resp. even) number. Note also that the minima of all nonempty π_{2i-1} must be odd, while the maxima of all nonempty π_{2i} must be even. Therefore, $\pi_{2i-1} \in \mathfrak{D}^1(2413, 3142)$ for all $i \geq 1$, whereas $\pi_{2i} \in \mathcal{D}^1(2413, 3142)$, i.e. $c(\pi_{2i}) \in \mathfrak{D}^1(2413, 3142)$, for all $i \geq 1$. Finally, the sum of the sizes of all π_i's is $2n - 2$.

Conversely, note that any permutation π constructed as above belongs to $\mathfrak{D}^1_{2n}(2413, 3142)$.

Let $a_n = |\mathfrak{D}^1_{2n}(2413, 3142)|$, and let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the ordinary generating function for $\{a_n\}_{n \geq 0}$, then the recursive structure of permutations in $\mathfrak{D}^1_{2n}(2413, 3142)$ described above implies that

$$F(x) = 1 + xF(x)^2 \frac{1}{1 - (F(x) - 1)},$$

or, equivalently,

$$(x + 1)F(x)^2 - 3F(x) + 2 = 0.$$

This implies the theorem. □

The following two results are obtained similarly to Theorem 3.11, and their proofs are left to the reader.

Theorem 3.12 *The ordinary generating function for* $|\mathfrak{D}^1_{2n}(1423, 4132)|$ *(*$n \geq 0$*) is given by*

$$G(x) = \frac{2-(1+x)C(x)}{2-x-(1+x)C(x)} = \frac{1-3x-(1+x)\sqrt{1-4x}}{1-3x-2x^2-(1+x)\sqrt{1-4x}}.$$

Theorem 3.13 *We have* $|\mathfrak{D}^1_{2n}(2413, 4132)| = |\mathfrak{D}^1_{2n}(1423, 3142)|$ *for* $n \geq 0$ *(i.e.* $(2413, 4132)$ *and* $(1423, 3142) = (c(4132), c(2413))$ *are* $\mathfrak{D}^1$*-Wilf-equivalent), and the ordinary generating function for each sequence is given by*

$$H(x) = \frac{1 + xC(x) - \sqrt{1 - xC(x) - 5x}}{2x(1 + C(x))}.$$

There are many directions in which to proceed further. We will only mention several.

One such direction is to complete the investigation of single forbidden patterns in $\mathfrak{D}^1_4$ and $\mathfrak{D}^2_4$, i.e. to find $|\mathfrak{D}^1_{2n}(\tau)|$ for $\tau = 2143, 3421, 4213$ and $|\mathfrak{D}^2_{2n}(\tau)|$ for $\tau = 2143, 4132$. Another is to combine the forbidden patterns of Section 3 with additional restrictions as in [8]. Yet another is to find the complete distribution for the number of occurrences of these patterns possibly combined with other restrictions, or to find equidistributed statistics on some of these restricted sets. Finally, the restrictions that define $\mathfrak{D}^1$ may be generalized to strings with repeated letters. It remains to be seen if a generalization to words is possible for $\mathfrak{D}^2$.

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