Math 301 **Homework 4** **Due 12/06/04**

For definition of a *conjugacy class* and notation $\mathrm{cl}(a)$, see Problem 3 on p. 89 of the textbook. Recall that elements are called *conjugate* if they belong to the same conjugacy class. Equivalently, for $x, y \in G$, y is conjugate to x in G if and only if there is $g \in G$ such that $y = gxg^{-1}$. Also recall that conjugacy is an equivalence relation.

Part A.

1. Prove that the multiplicative group of nonzero real numbers $\mathbb{R}^* = (\mathbb{R}\setminus\{0\}, \times)$ and the additive group of real numbers $\mathbb{R} = (\mathbb{R}, +)$ are not isomorphic. *Hint:* Assume there is a homomorphism $\phi : \mathbb{R}^* \to \mathbb{R}$. Which elements will have to be in $\phi^{-1}(0)$ and why? Use your result to show that ϕ cannot be an isomorphism.
2. Let G be a group and $x \in G$. Prove that $\mathrm{cl}(x)$ has exactly 1 element if and only if $x \in Z(G)$.
3. Let G be a finite group and suppose that $K \leq H \leq G$. Prove that $|G : K| = |G : H||H : K|$.

Part B.

1. Prove that A_n is generated by 3-cycles, i.e. every permutation in A_n can be expressed as a product of one or more 3-cycles. *Hint:* First express a permutation in A_n as a product of an even number of transpositions (2-cycles). Now use the fact that $(ab)(ab) = \varepsilon$, $(ab)(ac) = (acb)$ and $(ab)(cd) = (abc)(adc)$ for any distinct elements a, b, c, d.
2. Let G be a group, and let $N \lhd G$. (In other words, $xNx^{-1} = N$ for any $x \in G$.) Prove that N is a disjoint union of some conjugacy classes in G.
3. Define the *cycle structure* of a permutation as the list of the lengths of its cycles (except 1-cycles) in the cycle notation (with multiplicities) in non-increasing order. [E.g. (12)(385)(49) has cycle structure $(3, 2, 2)$.] Prove that any two permutations in S_n have the same cycle structure if and only if they are conjugate.

   Hint: This is not unlike what a change of basis does to matrices of linear operators, only here instead of basis vectors, we have elements $1, 2, \ldots, n$.
4. Assume the result of Problem 3 for this problem. Recall that $A_n \lhd S_n$ for any n. Which conjugacy cycles in S_5 are contained in A_5? How many elements does each of those conjugacy classes have?

Part C.

1. Let G be any group (finite or infinite) and suppose that $K \leq H \leq G$. Prove that $|G : K| = |G : H||H : K|$, i.e. either both sides are infinite, or both sides are finite and the equality above holds.

2. Again, assume the result of Problem B3 for this problem. Determine the conjugacy classes in A_5. (This is different from Problem 4 because now you can only conjugate by even permutations, i.e. y is conjugate to x in A_5 if and only if $y = gxg^{-1}$ for some *even* permutation g.) How many elements does each conjugacy class have?

3. Use the result of Problems B2 and C2 to prove that the group A_5 is *simple*, i.e. has no nontrivial normal subgroups (none except $\{\varepsilon\}$ and A_5). *Hint:* Prove that each conjugacy class in S_5 contained in A_5 is either a conjugacy class in A_5 or a disjoint union of two conjugacy classes of equal size in A_5. Also prove that (12345) and (12354) are not conjugate in A_5. Use the statement and your proof of Problem B4.

   Remark: This fact is used to prove that polynomial equations of degree 5 are not solvable by radicals. In other words, there is no general expression for zeros of an arbitrary polynomial of degree 5 using its coefficients, the arithmetic operations, and extraction of roots (radicals). So there is a quadratic formula, cubic formula, quartic formula, but no quintic formula. Similarly, one can show that the group A_n is simple for $n \geq 5$, so polynomial equations of degree $n \geq 5$ are not solvable by radicals. This is the fundamental result of Niels Henryk Abel (for $n = 5$) and Evariste Galois (for all $n \geq 5$).