Chapter 18

SUMMARY AND CONCLUSIONS

In this study, our primary interest has been to demonstrate, in a finite element method setting, the use of discretized sensitivities of geometric parameters to approximate the cost gradient in a flow optimization.

For the problems we have studied, we have seen that the finite element form of the discretized sensitivity equations are quite simple to formulate, inexpensive to solve, and practical to use. In particular, the discretized sensitivities can be computed at almost no cost at the end of the Newton iteration that solves the current state equation. These discretized sensitivities may then be used in place of the true sensitivities of the discrete solution. However, discretized sensitivities are only an approximation to the discrete sensitivities, and in our case, we expect shape parameter sensitivities to have an approximation error that depends on the discretization size h.

We have seen that the finite coefficient differences, (if properly adjusted in the case of shape parameters), are another valuable approximation to the discrete sensitivities. An advantage of finite coefficient differences is that higher accuracy can be achieved simply by decreasing the parameter perturbation $\Delta\alpha$, rather than by decreasing the mesh parameter h. A second advantage is that finite coefficient differences are good approximations whether or not the differentiation and discretization operators commute.

In contrast, the computation of the discrete sensitivities can be a daunting task. The sensitivity equation will include not merely the expected terms with a physical interpretaion, but also terms that arise from changes in the discretization process itself, such as changes in node placement and element shape. These terms have no physical meaning, and their proper treatment requires tedious effort.

Although the discretized sensitivities were inexpensive to compute and generally sufficiently accurate for our purposes, we have seen cases where the limited approximation power caused the optimization algorithm to fail. Since our horizontal velocities u were solved with an accuracy of $O(h^2)$, the spatial derivatives have an accuracy of $O(h)$, and this in turn limits the accuracy of our discretized sensitivities to $O(h)$. The same reasoning suggests that, for our particular problem and boundary conditions, if we improved the velocity approximation to order $O(h^n)$, the shape parameter sensitivities would be accurate to order $O(h^{n-1})$.

A number of successful computations were made with the discretized sensitivities. These computations involved shape optimization at a variety of Reynolds numbers. In cases where accuracy problems did not intrude, the discretized sensitivities did indeed show themselves to be an efficient tool for making linear estimates of the solution behavior and, in optimization, for approximating cost gradients.

On the other hand, the discretized sensitivities can be poor approximants of the discrete sensitivities, even in cases where the finite element method has done a good job approximating the solution of the continuous problem. The errors in such a case can best be analyzed by also computing the finite coefficient differences, or the finite cost gradient differences for comparison.

There is a reason we have explored discretized sensitivities and finite coefficient differences as means of approximating the discrete sensitivities. It is because the discrete sensitivities contain so much useful information; they provide the cost gradient in an optimization, they show when the cost functional is well conditioned; they give physical insight into the relationship between parameters and flow solutions; they are needed when using continuation to reach higher Reynolds number solutions; and they are useful in making linear estimates of the solution or cost function for nearby parameter values.

As the Reynolds number increases, and the finite element mesh parameter h is reduced, the size of the Jacobian increases, and the cost of evaluating and factoring it predominates the cost of computing a flow solution, and hence of carrying out an optimization. Our efficiency studies suggest that the effectiveness of some simple techniques, such as holding the Jacobian fixed during a particular Newton process, or only evaluating the Jacobian when the parameters have changed by more than some given tolerance.

A number of questions remain open as fruitful areas of further research:

- First, further study of the behavior of the discretized sensitivities at higher Reynolds numbers is needed, but computing limitations restricted our investigations to values no higher than $Re = 1000$.

- Secondly, it would be useful to have a carefully worked out estimate of the approximation error between the discrete sensitivities and the discretized sensitivities. In this connection, it is vital to examine the errors in the boundary condition for the discretized sensitivities, to quantify the influence these errors have on the computed discretized sensitivities, and to produce estimates of the behavior of this error as the mesh parameter h goes to zero.

- Finally, practical algorithmic procedures are needed for estimating the approximation error in the discretized sensitivities and reducing it when necessary, perhaps by local refinement of the grid or the local use of higher order elements.