Chapter 7

SENSITIVITIES FOR AN IMPLICIT PARAMETER

7.1 Introduction

In Chapter 6, we found that we could derive the sensitivity equations for an explicit variable by simple differentiation of the state equations. But if our parameter is *implicit*, that is, does not show up directly in the state equations, then simply formally differentiating the state equations will produce a homogeneous sensitivity system with zero solution, which can't be correct if the parameter does actually have some influence on the solution.

To derive the correct approach for computing such sensitivities, we need to return to the definition of a sensitivity as a change in the solution caused by changes in the parameter. In our examples, the influence of the parameter will show up in a nonzero source term in the boundary conditions.

The implicit parameters we will consider affect the geometry of the region. This means that we may have to make special adjustments when computing a partial derivative or making a finite difference quotient, at a fixed point in space. Special difficulties occur for points on the boundary, since even for small perturbations of the parameter, such a point is likely to move to the interior or exterior of the region. Moreover, the very structure of the discretization method will change, including the location of nodes, the shape of elements, and the definition of basis functions.

7.2 The Continuous α-Sensitivity Equations

We consider our standard problem of fluid flow in a channel with a bump, and attempt to derive the continuous sensitivity equations for a parameter that affects the shape of that bump. As usual, we denote this shape parameter by α, and note that while there might actually be several shape parameters in a particular problem, we will restrict our attention to a single one.

Let us suppose, then, that we have a particular parameter value α_0 and a solution $(u, v, p)(\alpha_0)$ to the corresponding flow problem. We will first consider the case of a point (x, y) lying in the interior of the flow region. We will do so by computing a second flow solution at a slightly altered parameter value, $\alpha_0 + \Delta\alpha$ and comparing the two solutions at (x, y). As long as the influence of the parameter α on the shape of the region is at least continuous, then because the point is contained strictly in the interior of the region, for all sufficiently small values of $\Delta\alpha$, the point will remain in the interior. Therefore, for both the unperturbed and perturbed parameter values, there will be a flow solution defined at the point. Hence, at such a point (x, y), we can define difference quotients of the flow solutions, and hence functions u_α, v_α and p_α which are the limits of those difference quotients. That is:

$$u_\alpha(x, y) \equiv \lim_{\Delta\alpha \to 0} \frac{u(\alpha_0 + \Delta\alpha) - u(\alpha_0)}{\Delta\alpha}. \tag{7.1}$$

These limit functions, the continuous sensitivities, will satisfy the homogeneous Oseen equations:

$$-(\frac{\partial^2 u_\alpha}{\partial x^2} + \frac{\partial^2 u_\alpha}{\partial y^2}) + Re\ (u_\alpha \frac{\partial u}{\partial x} + u\frac{\partial u_\alpha}{\partial x} + v_\alpha \frac{\partial u}{\partial y} + v\frac{\partial u_\alpha}{\partial y} + \frac{\partial p_\alpha}{\partial x}) = 0 \quad (7.2)$$

$$-(\frac{\partial^2 v_\alpha}{\partial x^2} + \frac{\partial^2 v_\alpha}{\partial y^2}) + Re\ (u_\alpha \frac{\partial v}{\partial x} + u\frac{\partial v_\alpha}{\partial x} + v_\alpha \frac{\partial v}{\partial y} + v\frac{\partial v_\alpha}{\partial y} + \frac{\partial p_\alpha}{\partial y}) = 0 \quad (7.3)$$

$$\frac{\partial u_\alpha}{\partial x} + \frac{\partial v_\alpha}{\partial y} = 0 \quad (7.4)$$

To complete the specification of our continuous sensitivity system, we have only to determine the form of the boundary conditions.

The pressure condition is easily disposed of. For every value of α, we require that $p(xmax, ymax) = 0$. Note that the values of $xmax$ and $ymax$ do not depend on the parameter; hence there is little doubt about how to take a difference quotient of this condition, and the limit operation results in the condition:

$$p_\alpha(xmax, ymax) = 0. \quad (7.5)$$

Now let us consider a point on a stationary wall. Both the horizontal and vertical velocities are zero here, for every value of α, and no pressure condition is applied. Hence the parameter can exert no influence whatsoever, and the appropriate boundary conditions are

$$u_\alpha = v_\alpha = 0 \text{ along the fixed walls.} \quad (7.6)$$

For similar reasons, we also easily determine the boundary condition:

$$\frac{\partial u_\alpha}{\partial x} = v_\alpha = 0 \text{ on the outflow boundary.} \quad (7.7)$$

The fact that the inflow velocity specification is always the same, regardless of the value of α means that we can also conclude that

$$u_\alpha = v_\alpha = 0 \text{ on the inflow boundary.} \quad (7.8)$$

At this point, we have almost specified a homogeneous problem. If the boundary condition along the bump were zero, then the solution to the Oseen problem would be the zero flow. Since apparently, this boundary condition will be the only way for the solution to be nonzero, the correct computation and accurate evaluation of this condition is crucial.

Recall that we have a function that defines the surface of the bump, of the form $y = Bump(x, \alpha)$. We now consider a fixed point (x, y) which, for the value $\alpha = \alpha_0$ lies on the bump surface. We want to analyze how the horizontal velocity at that point must be affected by changes in α. That means we must look at the value of $u(x, y)$ for solutions at nearby values of α. Of course, for any nearby value of α, the point (x, y) is probably no longer lying on the bump surface, and instead will probably lie outside the region, where we have no solution information, or in the interior, where we do. We much prefer the latter case. It seems reasonable to demand that our *Bump* function be specified so that, if a point (x, y) lies on the bump surface for the parameter value α_0, then it either remains on the bump surface for all values of α (the endpoints of the bump), or else, we can always find a perturbation $\Delta\alpha$ such that (x, y) lies strictly within $\Omega(\alpha)$ for every α strictly between α_0 and $\alpha_0 + \Delta\alpha$. Here, the perturbation $\Delta\alpha$ may in fact be negative.

Now we must try to compute an estimate for the value $u(x, y, \alpha_0 + \Delta\alpha)$ so that we can make our comparison. Since (x, y) was on the bump surface for $\alpha = \alpha_0$, and we have only changed α a small amount, it is reasonable to try to estimate the value of u at (x, y) by referring to its value at the nearby point $(x, Bump(x, \alpha_0 + \Delta\alpha))$. Since this reference point lies on the bump, we know that the value of u there is exactly zero:

$$u(x, Bump(x, \alpha_0 + \Delta\alpha), \alpha_0 + \Delta\alpha) = 0. \tag{7.9}$$

Using this information, we can use a Taylor estimate to deduce the value of u at (x, y):

$$u(x, y, \alpha_0 + \Delta\alpha) \quad = \quad u(x, Bump(x, \alpha_0 + \Delta\alpha), \alpha_0 + \Delta\alpha) \tag{7.10}$$

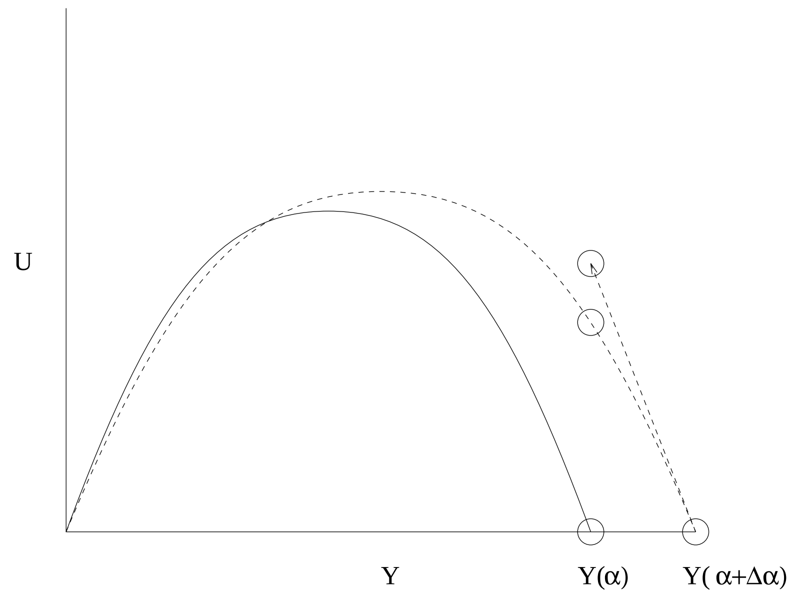

Figure 7.1: Estimating the solution at a moving point.
The original solution $u(x, y, \alpha_0)$ is the solid curve.
The perturbed solution $u(x, y, \alpha_0 + \Delta\alpha)$ is the dashed curve.
A two-term Taylor estimate approximates $u(x, \alpha_0, \alpha_0 + \Delta\alpha)$.

$$+ \quad \frac{\partial u}{\partial y}(x, \xi, \alpha_0 + \Delta\alpha)\ (y - Bump(x, \alpha_0 + \Delta\alpha)), \tag{7.11}$$

where ξ lies between $Bump(x, \alpha_0 + \Delta\alpha)$ and y. Using Equation (7.9) and the fact that $y = Bump(x, \alpha_0)$, our equation becomes:

$$u(x, y, \alpha_0 + \Delta\alpha) = \frac{\partial u}{\partial y}(x, \xi, \alpha_0 + \Delta\alpha)\ (Bump(x, \alpha_0) - Bump(x, \alpha_0 + \Delta\alpha)), \tag{7.12}$$

and if we assume that u is, at least locally, twice continuously differentiable in y, we can make an estimate using known quantities, with an error term:

$$\begin{aligned} u(x, y, \alpha_0 + \Delta\alpha) = & \frac{\partial u}{\partial y}(x, y, \alpha_0 + \Delta\alpha)\ (Bump(x, \alpha_0) - Bump(x, \alpha_0 + \Delta\alpha)) \\ & + O((Bump(x, \alpha_0) - Bump(x, \alpha_0 + \Delta\alpha))^2). \end{aligned} \tag{7.13}$$

Figure 7.1 suggests how we are going to estimate the value of u_α.

We now set up our difference quotient estimate, using the fact that $u(x, y, \alpha_0) = 0$:

$$u_\alpha(x, y, \alpha_0) = \frac{\partial u}{\partial \alpha}(x, y, \alpha_0) \tag{7.14}$$

$$= \lim_{\Delta\alpha \to 0} \frac{u(x,y,\alpha_0+\Delta\alpha) - u(x,y,\alpha_0)}{\Delta\alpha} \quad (7.15)$$

$$= \lim_{\Delta\alpha \to 0} \frac{\frac{\partial u}{\partial y}(x,y,\alpha_0+\Delta\alpha)(Bump(x,\alpha_0) - Bump(x,\alpha_0+\Delta\alpha)}{\Delta\alpha} \quad (7.16)$$

$$= -\frac{\partial u}{\partial y}(x,y,\alpha_0)\frac{\partial Bump}{\partial \alpha}(x,\alpha_0). \quad (7.17)$$

A second, and quicker, way to derive this same formula works by considering a point that *moves* with the surface of the bump. Such a point would have a varying y coordinate $y(\alpha) = Bump(x,\alpha)$. Then, for every value of α, the point lies on the bump and so the value of its horizontal velocity is zero. Therefore, we may assert:

$$\frac{Du}{D\alpha}(x, Bump(x,\alpha), \alpha) = \frac{\partial u}{\partial y}\frac{\partial Bump(x,\alpha)}{\partial \alpha} + \frac{\partial u}{\partial \alpha} = 0, \quad (7.18)$$

which immediately yields Equation (7.17).

Similar manipulations allow us to conclude that

$$v_\alpha(x, Bump(x,\alpha), \alpha) = -\frac{\partial v}{\partial y}\frac{Bump(x,\alpha)}{\partial \alpha}. \quad (7.19)$$

These two equations may be regarded as boundary conditions that are to replace the usual conditions on u and v on the bump. Our continuous α-sensitivity boundary conditions therefore have the form:

$$\begin{aligned}
u_\alpha(0,y) &= 0 \text{ on the inflow and walls;} \\
v_\alpha(0,y) &= 0 \text{ on the inflow, outflow and walls;} \\
u_\alpha(x,y) &= -\frac{\partial u}{\partial y}\frac{\partial Bump(x,\alpha)}{\partial \alpha} \text{ on the bump;} \\
v_\alpha(x,y) &= -\frac{\partial v}{\partial y}\frac{\partial Bump(x,\alpha)}{\partial \alpha} \text{ on the bump;} \\
\frac{\partial u_\alpha}{\partial x}(xmax,y) &= 0; \\
p_\alpha(xmax,ymax) &= 0.
\end{aligned}$$

These boundary conditions, together with the homogeneous Oseen equations, make up our continuous sensitivity system for the implicit parameter α.

7.3 The Discrete α-Sensitivity Equations

To derive the discrete α-sensitivity equations, we begin by writing out the discrete state equations (4.10), (4.11), and (4.12). We intend to differentiate these with respect to the shape parameter α. However, we cannot proceed by simply bringing the differentiation operator inside the integral; the case of α is special, because *the domain of integration depends on* α. We will try to emphasize that point in this section by dutifully designating the region as $\Omega(\alpha)$.

Thus, instead of the general rule:

$$\frac{\partial}{\partial \beta} \int_{\Omega} f(x,y,\beta)\; dxdy = \int_{\Omega} \frac{\partial f(x,y,\beta)}{\partial \beta}\; dxdy, \tag{7.20}$$

which holds for differentiable f and a domain of integration which does not depend on β, we must use the rule:

$$\frac{\partial}{\partial \alpha} \int_{\Omega(\alpha)} f(x,y,\alpha)\; dxdy \;=\; \int_{\Omega(\alpha)} \frac{\partial f(x,y,\alpha)}{\partial \alpha}\; dxdy \tag{7.21}$$

$$+ \int_{\Gamma(\alpha)} f(x(s),y(s),\alpha)\; \frac{d\hat{n}(s)}{d\alpha}\; ds. \tag{7.22}$$

Here, the expression $\frac{d\hat{n}(s)}{d\alpha}$ refers to changes in the outward unit normal vector along the boundary $\Gamma(\alpha)$. This formula tells us that a change in β causes changes to the original integral in two ways: the integrand f changes in the original integration region, and the region of integration itself Ω changes. We suggest this situation in Figure 7.2.

Note that this extra term comes about because of the fact that the finite element method uses an integral. If our discrete state equations had been derived from the finite difference method, no such extra term would arise.

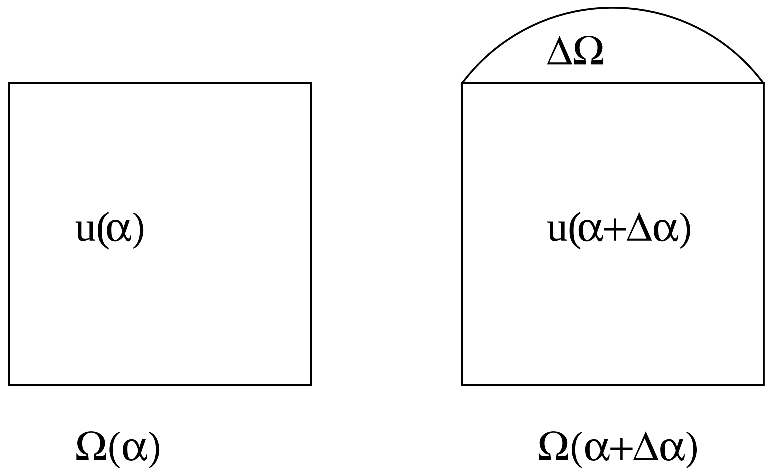

Figure 7.2: How an integral changes when the integrand and region both vary.
We compare changes in u over the original region.
We must also account for changes where the region expands.

With this formula as our guide, we may now proceed to derive the discrete α-sensitivity equations for the discretized finite element state equations. In this set of equations, we have suppressed the indexing of the basis functions w_i and q_i, since we are going to need to take partial derivatives of these functions:

$$\begin{aligned}
\int_{\Omega(\alpha)} (\frac{\partial u^h_\alpha}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u^h_\alpha}{\partial y}\frac{\partial w}{\partial y} + Re(u^h_\alpha \frac{\partial u^h}{\partial x} + u^h \frac{\partial u^h_\alpha}{\partial x} + v^h_\alpha \frac{\partial u^h}{\partial y} + v^h \frac{\partial u^h_\alpha}{\partial y} + \frac{\partial p^h_\alpha}{\partial x}) w) \; d\Omega \quad = \\
- \int_{\Omega(\alpha)} (\frac{\partial u^h}{\partial x}\frac{\partial w_\alpha}{\partial x} + \frac{\partial u^h}{\partial y}\frac{\partial w_\alpha}{\partial y} + Re(u^h \frac{\partial u^h}{\partial x} + u^h \frac{\partial u^h}{\partial x} + v^h \frac{\partial u^h}{\partial y} + v^h \frac{\partial u^h}{\partial y} + \frac{\partial p^h}{\partial x}) w_\alpha) \; d\Omega \\
- \int_{\Gamma(\alpha)} (\frac{\partial u^h}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial u^h}{\partial y}\frac{\partial w}{\partial y} + Re(u^h \frac{\partial u^h}{\partial x} + v^h \frac{\partial u^h}{\partial y} + \frac{\partial p^h}{\partial x}) w) \; \frac{d\hat{n}}{d\alpha} \; d\Gamma
\end{aligned} \tag{7.23}$$

$$\begin{aligned}
\int_{\Omega(\alpha)} (\frac{\partial v^h_\alpha}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v^h_\alpha}{\partial y}\frac{\partial w}{\partial y} + Re(u^h_\alpha \frac{\partial v^h}{\partial x} + u^h \frac{\partial v^h_\alpha}{\partial x} + v^h_\alpha \frac{\partial v^h}{\partial y} + v^h \frac{\partial v^h_\alpha}{\partial y} + \frac{\partial p^h_\alpha}{\partial y}) w) \; d\Omega \\
= - \int_{\Omega(\alpha)} (\frac{\partial v^h}{\partial x}\frac{\partial w_\alpha}{\partial x} + \frac{\partial v^h}{\partial y}\frac{\partial w_\alpha}{\partial y} + Re(u^h \frac{\partial v^h}{\partial x} + u^h \frac{\partial v^h}{\partial x} + v^h \frac{\partial v^h}{\partial y} + v^h \frac{\partial v^h}{\partial y} + \frac{\partial p^h}{\partial y}) w_\alpha) \; d\Omega \\
- \int_{\Gamma(\alpha)} (\frac{\partial v^h}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v^h}{\partial y}\frac{\partial w}{\partial y} + Re(u^h \frac{\partial v^h}{\partial x} + v^h \frac{\partial v^h}{\partial y} + \frac{\partial p^h}{\partial y}) w) \frac{d\hat{n}}{d\alpha} \; d\Gamma
\end{aligned} \tag{7.24}$$

$$\int_{\Omega(\alpha)} (\frac{\partial u^h_\alpha}{\partial x} + \frac{\partial v^h_\alpha}{\partial y}) q \; d\Omega = - \int_{\Omega(\alpha)} (\frac{\partial u^h}{\partial x} + \frac{\partial v^h}{\partial y}) q_\alpha \; d\Omega - \int_{\Gamma(\alpha)} (\frac{\partial u^h}{\partial x} + \frac{\partial v^h}{\partial y}) q \frac{d\hat{n}}{d\alpha} \; d\Gamma$$

(7.25)

The complicated form of these equations may be something of a surprise. Notice, though, that the left hand side, involving the unknowns, has the same form as the discrete sensitivity systems for Re and λ. The complications show up only on the right hand side. Aside from the new boundary integral over Γ, something else has entered the equations: partial derivatives of basis functions w and q with respect to the parameter. This is a peculiarly unwelcome development since it will require a careful analysis of the dependence of the basis functions upon the node locations, and their dependence, in turn, on the value of α.

The boundary conditions are derived in the usual way, that is, by differentiating them, except that the condition on the bump must be deduced as described earlier for the continuous sensitivity equation. The result is:

$$\begin{aligned}
u^h_\alpha(x,y) &= 0 \text{ at inflow and wall velocity nodes;}\\
v^h_\alpha(x,y) &= 0 \text{ at inflow, outflow and wall velocity nodes;}\\
u^h_\alpha(x,y) &= -\frac{\partial u^h}{\partial y}\frac{\partial Bump(x,\alpha)}{\partial \alpha} \text{ at bump velocity nodes;}\\
v^h_\alpha(x,y) &= -\frac{\partial v^h}{\partial y}\frac{\partial Bump(x,\alpha)}{\partial \alpha} \text{ at bump velocity nodes;}\\
p_\alpha(xmax,ymax) &= 0 \text{ at the upper right pressure node.}
\end{aligned}$$

For comparison, we will now discretize the continuous sensitivity equation, that is, reverse the order of the operations of discretization and differentiation. We will see that the resulting equations are significantly different.

7.4 The Discretized α-Sensitivity Equations

To derive the discretized sensitivity equations, we must refer to the continuous sensitivity equations, (7.2), (7.3), and (7.4), and apply the finite element discretization:

$$\begin{aligned} \int_{\Omega(\alpha)} (\frac{\partial (u_\alpha)^h}{\partial x}\frac{\partial w_i}{\partial x} + \frac{\partial (u_\alpha)^h}{\partial y}\frac{\partial w_i}{\partial y} & \\ +Re((u_\alpha)^h \frac{\partial u^h}{\partial x} + u^h \frac{\partial (u_\alpha)^h}{\partial x} + (v_\alpha)^h \frac{\partial u^h}{\partial y} + v^h \frac{\partial (u_\alpha)^h}{\partial y} + \frac{\partial (p_\alpha)^h}{\partial x}) w_i)\ d\Omega &= 0 \end{aligned} \tag{7.26}$$

$$\begin{aligned} \int_{\Omega(\alpha)} (\frac{\partial (v_\alpha)^h}{\partial x}\frac{\partial w_i}{\partial x} + \frac{\partial (v_\alpha)^h}{\partial y}\frac{\partial w_i}{\partial y} & \\ +Re((u_\alpha)^h \frac{\partial v^h}{\partial x} + u^h \frac{\partial (v_\alpha)^h}{\partial x} + (v_\alpha)^h \frac{\partial v^h}{\partial y} + v^h \frac{\partial (v_\alpha)^h}{\partial y} + \frac{\partial (p_\alpha)^h}{\partial y}) w_i)\ d\Omega &= 0 \end{aligned} \tag{7.27}$$

$$\int_{\Omega(\alpha)} (\frac{\partial (u_\alpha)^h}{\partial x} + \frac{\partial (v_\alpha)^h}{\partial y}) q_i\ d\Omega = 0 \tag{7.28}$$

The boundary conditions are straightforwardly derived, except that the conditions on the bump are in terms of spatial derivatives of the exact solution u, which we do not know. Therefore, we must approximate those conditions using spatial derivatives of the discretized solution u^h:

$$\begin{aligned} (u_\alpha)^h(x,y) &= 0 \text{ at inflow and wall velocity nodes;} \\ (v_\alpha)^h(x,y) &= 0 \text{ at inflow, outflow and wall velocity nodes;} \\ (u_\alpha)^h(x,y) &= -\frac{\partial u}{\partial y}\frac{\partial Bump(x,\alpha)}{\partial \alpha} \\ &\approx -\frac{\partial u^h}{\partial y}\frac{\partial Bump(x,\alpha)}{\partial \alpha} \end{aligned}$$

at bump velocity nodes;

$$(v_\alpha)^h(x,y) = -\frac{\partial v}{\partial y}\frac{\partial Bump(x,\alpha)}{\partial \alpha}$$

$$\approx \quad -\frac{\partial v^h}{\partial y}\frac{\partial Bump(x,\alpha)}{\partial \alpha} \text{ at bump velocity nodes;}$$

$$(p_\alpha)^h(xmax, ymax) \quad = \quad 0 \text{ at the upper right pressure node.}$$

This discretized sensitivity system contrasts strongly with the discrete sensitivity system. In particular, the right hand sides are zero, whereas the right hand sides of the discrete sensitivity system exhibited many terms that will require extensive calculations.

Nonetheless, note that both systems have the same form on the left hand side. Thus, we might say that two systems agree on the operator, but differ on the right hand side.

Finally we note that, for the discretized sensitivity equations, the boundary conditions along the bump are a primary source of error. To apply the boundary condition exactly, we need to know the partial derivatives of the *true solution* u and v. In fact, we only have knowledge of the discretized solution variables u^h and v^h, and our approximation is made worse by the fact that we are working at points on the boundary, and because we are approximating a spatial derivative rather than a state variable. Roughly speaking, we have seen that the true velocities u are approximated by the Taylor Hood finite element method with an error which we estimate to be of order $O(h^2)$. We said that this meant that a spatial derivative like $\frac{\partial u}{\partial y}$ would be approximated with an error of $O(h)$. This means that there is an error of order $O(h)$ in our specification of the boundary condition for the discretized shape sensitivities u^h_α.

These problems do not occur for the discrete sensitivity equations, because there the boundary conditions are given in terms of the discrete solution, which is known exactly.

7.5 Exact Example: Shape Sensitivity of Poiseuille Flow

Let us try to get our bearings by working with the simple case of Poiseuille flow.

Since we can only produce exact solutions of the Poiseuille flow when the channel is rectangular, we don't want to insert a curved bump into the channel. Instead, our parameter α will move the upper wall up and down, as a whole. Thus, the value of α will be the coordinate of the upper wall.

We assume an inflow profile of the form:

$$Inflow(y, \alpha, \lambda) = \lambda y(\alpha - y). \tag{7.29}$$

For any $0 < \alpha$, we can write out the exact continuous state solution at every point in the domain:

$$u(x, y, \alpha) = \lambda y(\alpha - y) \tag{7.30}$$

$$v(x, y, \alpha) = 0 \tag{7.31}$$

$$p(x, y, \alpha) = 2\lambda(xmax - x)/Re \tag{7.32}$$

and we can write the sensitivities with respect to the implicit shape parameter α:

$$u_\alpha(x, y, \alpha) = -\lambda y \tag{7.33}$$

$$v_\alpha(x, y, \alpha) = 0 \tag{7.34}$$

$$p_\alpha(x, y, \alpha) = 0 \tag{7.35}$$

Note that, for this special problem, the $Inflow$ function also has a dependence on α, and that we will make use of the fact that for this simple problem,

$$\frac{\partial Bump}{\partial \alpha} = 1. \tag{7.36}$$

The continuous α-sensitivity equations for this problem are:

$$-(\frac{\partial^2 u_\alpha}{\partial x^2} + \frac{\partial^2 u_\alpha}{\partial y^2}) + Re(u_\alpha \frac{\partial u}{\partial x} + u\frac{\partial u_\alpha}{\partial x} + v_\alpha \frac{\partial u}{\partial y} + v\frac{\partial u_\alpha}{\partial y} + \frac{\partial p_\alpha}{\partial x}) = 0 \tag{7.37}$$

$$-(\frac{\partial^2 v_\alpha}{\partial x^2} + \frac{\partial^2 v_\alpha}{\partial y^2}) + Re(u_\alpha \frac{\partial v}{\partial x} + u\frac{\partial v_\alpha}{\partial x} + v_\alpha \frac{\partial v}{\partial y} + v\frac{\partial v_\alpha}{\partial y} + \frac{\partial p_\alpha}{\partial y}) = 0 \tag{7.38}$$

$$\frac{\partial u_\alpha}{\partial x} + \frac{\partial v_\alpha}{\partial y} = 0 \tag{7.39}$$

with the boundary conditions:

$$
\begin{aligned}
u_\alpha(0,y) &= \frac{\partial Inflow(y,\alpha,\lambda)}{\partial \alpha} \\
&= \lambda y; \\
v_\alpha(0,y) &= 0; \\
u_\alpha(x,y) &= -\frac{\partial u}{\partial y} \\
&= -\lambda y \text{ on the upper wall;} \\
v_\alpha(x,y) &= -\frac{\partial v}{\partial y} = 0 \text{ on the upper wall;} \\
u_\alpha(x,y) &= v_\alpha(x,y) = 0 \text{ on the lower wall;} \\
\frac{\partial u_\alpha}{\partial x}(xmax,y) &= 0 \text{ on the outflow;} \\
v_\alpha(xmax,y) &= 0 \text{ on the outflow;} \\
p_\alpha(xmax,ymax) &= 0.
\end{aligned}
$$

We can easily verify that this system is satisfied by the sensitivities that we would get by differentiating the exact solution:

$$u_\alpha(x,y,\alpha) = \lambda y \tag{7.40}$$

$$v_\alpha(x,y,\alpha) = 0 \tag{7.41}$$

$$p_\alpha(x,y,\alpha) = 0 \tag{7.42}$$

Thus, we have shown that we can derive the continuous sensitivity equations for the Poiseuille flow with a moving wall, and compute values for the sensitivities which are equal to the known, exact values.

We should note that the shape parameter required us not only to add a new boundary condition along the moving shape, but also had an effect on the form of the inflow boundary condition. This happened because the shape parameter affected the region where the inflow

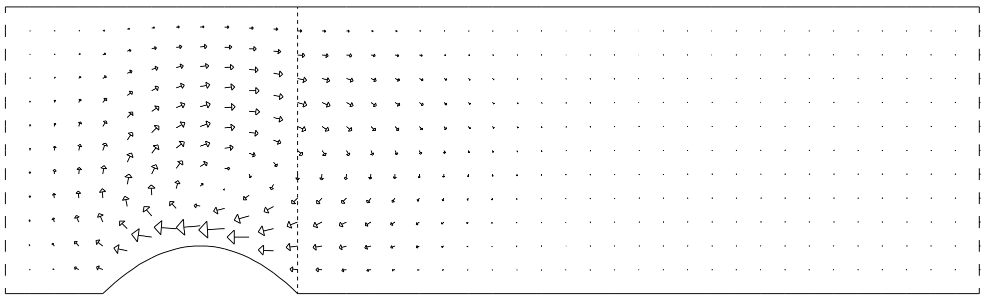

Figure 7.3: Discretized velocity α-sensitivity.

boundary condition was specified. Such secondary effects of a shape parameter may be hard to foresee or calculate in general.

7.6 Computational Example: Flow Past a Bump

If we return to the three-parameter problem already discussed in section 6.6, we may now compute the discretized flow sensitivities with respect to the parameter α that affects the flow variables implicitly by determining the shape of the bump that intrudes into the flow region.

Now we can, if we wish, plot the discretized velocity sensitivity field as though it were a physical quantity, as in Figure 7.3. In this case, a plot vector drawn at a particular point represents the relative change that would apply to the current velocity at that point, with a unit change in α. The discretized sensitivity velocity vectors don't behave like a physical flow: they satisfy the Oseen equations rather than the Navier Stokes equations. However, the same continuity equation appears in both sets of equations, and we can see directly from the plot that the continuity equation seems to be satisfied: the vector quantity represented by the plotted arrows seems to satisfy the rule that "what comes in goes out".

Table 7.1: Finite difference check of discretized α-sensitivities.

Variable	$\|(u_\alpha)^h\|_\infty$	$\|\Delta(u^h)/\Delta(\alpha)\|_\infty$	$\|\text{Difference}\|_\infty$
U	0.6541	0.8046	1.5E-01
V	0.2709	0.2702	1.2E-01
P	0.4709	0.4378	2.3E-01

We can glean other useful information from this figure. The plot shows that the influence of the shape parameter is restricted to the area near the bump, generating a sort of "whorl". Further downstream, the influence is negligible. This means that, unlike the inflow parameter, measurements of the effect of the shape parameter should be made near to the bump. We will see the importance of this fact later, when we try to use a profile line that is too far downstream from the bump.

7.7 Comparison of Finite Differences and Discretized Sensitivities

We return to the three-parameter problem discussed in Section 6.7. We would like to make a similar comparison, between the discretized sensitivities $(u_\alpha)^h$ and estimates of the influence of α made using finite differences. As in the previous table, we will simply compare the vector of finite element coefficients for $(u_\alpha)^h$ with the vector of differences in the finite element coefficients for u at α and at $\alpha + \Delta\alpha$, divided by $\Delta\alpha$. Table 7.1 reports the maximum norm of these two vectors, and of their difference. As in the previous Tables 6.1 and 6.2, our mesh parameter is $h = 0.25$. The perturbation $\Delta(\alpha)$ was computed using a formula similar to Equation (6.46).

The discrepancy here is startling, particularly when compared to the near perfect agreement obtained for the λ and Re sensitivities. There are actually several factors to investigate. First, we should already be aware that we are using discretized sensitivities, rather than discrete sensitivities. Secondly, as we will discover, our finite difference results are comparing the same coefficient at different values of α. If α were not a shape variable, then the two values of the coefficient would both be associated with the same fixed spatial value. But because we regrid the region when we compute the solution at $\alpha + \Delta\alpha$, our two coefficient values are actually associated with different positions, which is not a proper approximation to the *partial* derivative of $u^h(x, y, \alpha)$, which holds x and y fixed.

The proper computation of finite difference estimates of the sensitivities, and their comparison to discretized sensitivities, is a complicated matter that we will analyze carefully in the next chapter. At the end of that chapter, we will again make a comparison chart like Table 7.1, but we will have better agreement, and be able to explain what is going on.