# Chapter 6

# SENSITIVITIES FOR AN EXPLICIT PARAMETER

## 6.1 Introduction

We now consider the computation of sensitivities for a Navier Stokes problem. We are interested in both the continuous and discrete problems, and we would like an idea of the form of the resulting sensitivity equations, and perhaps an idea of what a sensitivity "looks like" for such a problem.

We consider how, starting with a general set of continuous Navier Stokes equations, we may derive the sensitivity equations with respect to the Reynolds number $Re$, or a parameter $\lambda$ that controls the strength of the inflow. We derive the corresponding discrete sensitivity equations for the discretized Navier Stokes equations. We then discretize the continuous sensitivity equation, to derive the equation for the discretized sensitivities.

We make a simple check of these calculations for the Poiseuille flow solution, computing the continuous, discrete and discretized sensitivities directly, and plugging them into the derived equations.

We display pictures of some computations of sensitivities for a discrete problem involving fluid flow in a channel with a bump. We also exhibit a table showing excellent agreement between the discrete sensitivities and the discretized sensitivities.

## 6.2 *Re*-Sensitivity Equations

Let us suppose that our parameter is the Reynolds number, $Re$, and that for a particular value of $Re_0$, we have a state solution $(u_0, v_0, p_0)$. We assume that there is an open neighborhood of $Re_0$, so that, for any $Re$ in that neighborhood, and for any $(x, y)$ in the flow region, the mixed derivatives of the state variables with respect to $Re$ and either $x$ or $y$ exist and are continuous. Then we can apply Clairaut's theorem, which allows us to interchange the order of differentiations with respect to $Re$ and either $x$ or $y$.

The Reynolds number appears explicitly, and in a simple way, in the horizontal and vertical momentum equations. We differentiate the continuous state equations with respect to $Re$, and where necessary, interchange so that differentiation of state variables with respect to $Re$ happens first. We then abbreviate the expression $\frac{\partial u}{\partial Re}$ as $u_{Re}$.

With these assumptions, we arrive at the following set of *continuous Re-sensitivity equations*:

$$
\begin{aligned}
-(\frac{\partial^2 u_{Re}}{\partial x^2} + \frac{\partial^2 u_{Re}}{\partial y^2}) \quad &+ \quad Re(u_{Re}\frac{\partial u}{\partial x} + u\frac{\partial u_{Re}}{\partial x} + v_{Re}\frac{\partial u}{\partial y} + v\frac{\partial u_{Re}}{\partial y} + \frac{\partial p_{Re}}{\partial x}) \\
&= \quad -(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}) \qquad (6.1) \\
-(\frac{\partial^2 v_{Re}}{\partial x^2} + \frac{\partial^2 v_{Re}}{\partial y^2}) \quad &+ \quad Re(u_{Re}\frac{\partial v}{\partial x} + u\frac{\partial v_{Re}}{\partial x} + v_{Re}\frac{\partial v}{\partial y} + v\frac{\partial v_{Re}}{\partial y} + \frac{\partial p_{Re}}{\partial y}) \\
&= \quad -(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y}) \qquad (6.2) \\
\frac{\partial u_{Re}}{\partial x} + \frac{\partial v_{Re}}{\partial y} \quad &= \quad 0 \qquad (6.3)
\end{aligned}
$$

with differentiated boundary conditions:

$$v_{Re}(x, y) \quad = \quad 0 \text{ along the boundary;}$$

$$\begin{aligned} u_{Re}(x,y) &= 0 \text{ along the inflow boundary, the walls, and the bump;} \\ \frac{\partial u_{Re}}{\partial x}(xmax,y) &= 0 \text{ on the outflow;} \\ p_{Re}(xmax,ymax) &= 0. \end{aligned}$$

To get the discrete $Re$-sensitivity equations, we recall the discrete state equations:

$$\int_\Omega (\frac{\partial u^h}{\partial x}\frac{\partial w_i}{\partial x} + \frac{\partial u^h}{\partial y}\frac{\partial w_i}{\partial y} \quad + \quad Re(u^h \frac{\partial u^h}{\partial x} + v^h \frac{\partial u^h}{\partial y} + \frac{\partial p^h}{\partial x}) w_i) \, d\Omega = 0 \tag{6.4}$$

$$\int_\Omega (\frac{\partial v^h}{\partial x}\frac{\partial w_i}{\partial x} + \frac{\partial v^h}{\partial y}\frac{\partial w_i}{\partial y} \quad + \quad Re(u^h \frac{\partial v^h}{\partial x} + v^h \frac{\partial v^h}{\partial y} + \frac{\partial p^h}{\partial y}) w_i) \, d\Omega = 0 \tag{6.5}$$

$$\int_\Omega (\frac{\partial u^h}{\partial x} + \frac{\partial v^h}{\partial y}) q_i \, d\Omega \quad = \quad 0 \tag{6.6}$$

and boundary conditions:

$$v^h(x,y) \quad = \quad 0 \text{ at boundary velocity nodes;} \tag{6.7}$$

$$u^h(0,y) \quad = \quad Inflow^h(y,\lambda) \text{ at inflow velocity nodes;} \tag{6.8}$$

$$u^h(x,y) \quad = \quad 0 \text{ at velocity nodes on the walls and the bump;} \tag{6.9}$$

$$p^h(xmax,ymax) \quad = \quad 0 \text{ for the upper right pressure node.} \tag{6.10}$$

If we now differentiate these equations with respect to $Re$, bringing the differentiation inside the integral signs and interchanging differentiations where necessary, we arrive at the *discrete $Re$-sensitivity equations*:

$$\begin{aligned} \int_\Omega (\frac{\partial u^h_{Re}}{\partial x}\frac{\partial w_i}{\partial x} + \frac{\partial u^h_{Re}}{\partial y}\frac{\partial w_i}{\partial y} \quad &+ \quad Re(u^h_{Re} \frac{\partial u^h}{\partial x} + u^h \frac{\partial u^h_{Re}}{\partial x} + v^h_{Re} \frac{\partial u^h}{\partial y} + v^h \frac{\partial u^h_{Re}}{\partial y} + \frac{\partial p^h_{Re}}{\partial x}) w_i) \, d\Omega \\ &= \quad \int_\Omega -(u^h \frac{\partial u^h}{\partial x} + v^h \frac{\partial u^h}{\partial y} + \frac{\partial p^h}{\partial x}) w_i \, d\Omega \end{aligned} \tag{6.11}$$

$$\begin{aligned} \int_\Omega (\frac{\partial v^h_{Re}}{\partial x}\frac{\partial w_i}{\partial x} + \frac{\partial v^h_{Re}}{\partial y}\frac{\partial w_i}{\partial y} \quad &+ \quad Re(u^h_{Re} \frac{\partial v^h}{\partial x} + u^h \frac{\partial v^h_{Re}}{\partial x} + v^h_{Re} \frac{\partial v^h}{\partial y} + v^h \frac{\partial v^h_{Re}}{\partial y} + \frac{\partial p^h_{Re}}{\partial y}) w_i) \, d\Omega \\ &= \quad \int_\Omega -(u^h \frac{\partial v^h}{\partial x} + v^h \frac{\partial v^h}{\partial y} + \frac{\partial p^h}{\partial y}) w_i \, d\Omega \end{aligned} \tag{6.12}$$

$$\int_\Omega (\frac{\partial u^h_{Re}}{\partial x} + \frac{\partial v^h_{Re}}{\partial y}) q_i \, d\Omega = 0 \quad (6.13)$$

with boundary conditions:

$$\begin{aligned} v^h_{Re}(x,y) &= 0 \text{ for boundary velocity nodes;} \\ u^h_{Re}(x,y) &= 0 \text{ for velocity nodes on the inflow, walls, and bump;} \\ p^h_{Re}(xmax, ymax) &= 0 \text{ at the upper right pressure node.} \end{aligned}$$

On the other hand, if we apply the finite element method to the sensitivity equations for the continuous system, we get the equations for the discretized $Re$-sensitivities:

$$\begin{aligned} \int_\Omega (\frac{\partial (u_{Re})^h}{\partial x}\frac{\partial w_i}{\partial x} + \frac{\partial (u_{Re})^h}{\partial y}\frac{\partial w_i}{\partial y} &+ Re((u_{Re})^h \frac{\partial u^h}{\partial x} + u^h \frac{\partial (u_{Re})^h}{\partial x} + (v_{Re})^h \frac{\partial u^h}{\partial y} + v^h \frac{\partial (u_{Re})^h}{\partial y} + \frac{\partial (p_{Re})^h}{\partial x}) w_i) \, d \\ &= \int_\Omega -(u^h \frac{\partial u^h}{\partial x} + v^h \frac{\partial u^h}{\partial y} + \frac{\partial p^h}{\partial x}) w_i \, d\Omega \quad (6.1 \\ \int_\Omega (\frac{\partial (v_{Re})^h}{\partial x}\frac{\partial w_i}{\partial x} + \frac{\partial (v_{Re})^h}{\partial y}\frac{\partial w_i}{\partial y} &+ Re((u_{Re})^h \frac{\partial v^h}{\partial x} + u^h \frac{\partial (v_{Re})^h}{\partial x} + (v_{Re})^h \frac{\partial v^h}{\partial y} + v^h \frac{\partial (v_{Re})^h}{\partial y} + \frac{\partial (p_{Re})^h}{\partial y}) w_i) \, d \\ &= \int_\Omega -(u^h \frac{\partial v^h}{\partial x} + v^h \frac{\partial v^h}{\partial y} + \frac{\partial p^h}{\partial y}) w_i \, d\Omega \quad (6.1 \\ \int_\Omega (\frac{\partial (u_{Re})^h}{\partial x} + \frac{\partial (v_{Re})^h}{\partial y}) q_i \, d\Omega &= 0 \quad (6.1 \end{aligned}$$

with boundary conditions:

$$\begin{aligned} (v_{Re})^h(x,y) &= 0 \text{ for boundary velocity nodes;} \\ (u_{Re})^h(x,y) &= 0 \text{ for velocity nodes on the inflow, walls, and bump;} \\ (p_{Re})^h(xmax, ymax) &= 0 \text{ at the upper right pressure node.} \end{aligned}$$

A glance at the discrete sensitivity equations and the discretized sensitivity equations shows that we have arrived at the same system; hence, for the $Re$ parameter, the discrete and discretized sensitivities are identical.

## 6.3 λ-Sensitivity Equations

We now derive the sensitivity equations for $\lambda$, a parameter that influences the strength of the *Inflow* function. Exactly as for *Re*, we proceed to derive the *continuous λ-sensitivity equations*:

$$-(\frac{\partial^2 u_\lambda}{\partial x^2}+\frac{\partial^2 u_\lambda}{\partial y^2})+Re(u_\lambda\frac{\partial u}{\partial x}+u\frac{\partial u_\lambda}{\partial x}+v_\lambda\frac{\partial u}{\partial y}+v\frac{\partial u_\lambda}{\partial y}+\frac{\partial p_\lambda}{\partial x}) = 0 \quad (6.17)$$

$$-(\frac{\partial^2 v_\lambda}{\partial x^2}+\frac{\partial^2 v_\lambda}{\partial y^2})+Re(u_\lambda\frac{\partial v}{\partial x}+u\frac{\partial v_\lambda}{\partial x}+v_\lambda\frac{\partial v}{\partial y}+v\frac{\partial v_\lambda}{\partial y}+\frac{\partial p_\lambda}{\partial y}) = 0 \quad (6.18)$$

$$\frac{\partial u_\lambda}{\partial x}+\frac{\partial v_\lambda}{\partial y} = 0 \quad (6.19)$$

with differentiated boundary conditions:

$$\begin{aligned}
v_\lambda(x,y) &= 0 \text{ along the boundary;}\\
u_\lambda(x,y) &= \frac{\partial Inflow(y,\lambda)}{\partial \lambda} \text{ along the inflow boundary;}\\
u_\lambda(x,y) &= 0 \text{ along the walls, and the bump;}\\
\frac{\partial u_\lambda}{\partial x}(xmax,y) &= 0 \text{ on the outflow;}\\
p_\lambda(xmax,ymax) &= 0.
\end{aligned}$$

As for the *Re* parameter, we are about to discover that the discrete sensitivity equations and the discretized sensitivity equations are identical. Therefore, we will only write out one set of equations.

Applying the finite element method to the continuous sensitivity equations, we get the *discretized λ-sensitivity equation*:

$$\begin{aligned}
\int_\Omega(\frac{\partial(u_\lambda)^h}{\partial x}\frac{\partial w_i}{\partial x} &+ \frac{\partial(u_\lambda)^h}{\partial y}\frac{\partial w_i}{\partial y}\\
&+ Re((u_\lambda)^h\frac{\partial u^h}{\partial x}+u^h\frac{\partial(u_\lambda)^h}{\partial x}+(v_\lambda)^h\frac{\partial u^h}{\partial y}+v^h\frac{\partial(u_\lambda)^h}{\partial y}+\frac{\partial(p_\lambda)^h)}{\partial x})w_i)\,d\Omega = 0 \quad (6.20)\\
\int_\Omega(\frac{\partial(v_\lambda)^h}{\partial x}\frac{\partial w_i}{\partial x} &+ \frac{\partial(v_\lambda)^h}{\partial y}\frac{\partial w_i}{\partial y}
\end{aligned}$$

$$+ \quad Re((u_\lambda)^h \frac{\partial v^h}{\partial x} + u^h \frac{\partial (v_\lambda)^h}{\partial x} + (v_\lambda)^h \frac{\partial v^h}{\partial y} + v^h \frac{\partial (v_\lambda)^h}{\partial y} + \frac{\partial (p_\lambda)^h)}{\partial y})w_i) \, d\Omega = 0 \tag{6.21}$$

$$\int_\Omega (\frac{\partial (u_\lambda)^h}{\partial x} \quad + \quad \frac{\partial (v_\lambda)^h}{\partial y})q_i \, d\Omega = 0 \tag{6.22}$$

with boundary conditions:

$$\begin{aligned}
(v_\lambda)^h(x,y) &= 0 \text{ for boundary velocity nodes;} \\
(u_\lambda)^h(0,y) &= \frac{\partial Inflow^h(y,\lambda)}{\partial \lambda} \text{ for outflow velocity nodes;} \\
(u_\lambda)^h(x,y) &= 0 \text{ for velocity nodes on the walls and bump;} \\
(p_\lambda)^h)(xmax,ymax) &= 0 \text{ for the upper right pressure node.}
\end{aligned}$$

Again, because the discrete sensitivity equations are identical in form to the discretized sensitivity equations, we may assert the equality of the discrete sensitivities $u^h_\lambda$ and the discretized sensitivities $(u_\lambda)^h$.

## 6.4 Discretization and Differentiation May Commute

For both the Reynolds and inflow parameters, we reached the same set of equations by two distinct methods, starting from the continous state equations:

- Discretize to determine the discrete state equations, then differentiate to determine the discrete sensitivity equations;
- Differentiate to determine the continuous sensitivity equations, then discretize to determine the discretized sensitivity equations.

The commutation of these two operations, if it occurs, is quite convenient. It allows us to do the sensitivity operations in a continuous space, where they are easy to perform, and to apply the discretization last, which means we can rely on a variety of theorems for the finite element method to show convergence of the discretized sensitivities to the continuous sensitivities. Thus, if we have commutation, then at the same time, the discretized sensitivities are the exact partial derivatives of the discrete solution with respect to the parameter (useful for various calculations), *and* there are the finite element approximants to the continuous sensitivities (which is useful for convergence analysis).

In fact, we can show that this commutation occurs between the application of the finite element method and differentiation with respect to parameters, as long as the parameters don't influence the region, the basis functions, or other items that would affect the finite element implementation:

**Theorem 6.1 (Commutation for Non-Geometric Parameters)** *Consider a system of differential equations $F(u,\beta) = 0$ with boundary conditions $G(u,\beta) = 0$ for the unknown function $u(x,y)$. Suppose that this system is suitable for treatment by the finite element method. Suppose that the implicit function theorem may be applied, so that $u(x,y)$ may be regarded as a continuously differentiable function of $\beta$, which we write $u(x,y,\beta)$. Suppose that $F$ and $G$ are continuously differentiable in their arguments $u$ and $\beta$. Suppose that the operations of differentiation and discretization commute on both $F$ and $G$, so that we may write*

$$F_u^h = (F^h)_u = (F_u)^h \tag{6.23}$$

$$F_\beta^h = (F^h)_\beta = (F_\beta)^h \tag{6.24}$$

$$G_u^h = (G^h)_u = (G_u)^h \tag{6.25}$$

$$G_\beta^h = (G^h)_\beta = (G_\beta)^h \tag{6.26}$$

*Suppose, finally, that the parameter $\beta$ has no influence on the shape of the domain $\Omega$, the value of basis functions, or on other such features of the finite element discretization. Then the discretized sensitivity equations are identical to the sensitivity equations for the discretized* *system.*

PROOF: The continuous sensitivity equations will have the form:

$$F_u(u,\beta)u_\beta + F_\beta(u,\beta) = 0 \tag{6.27}$$

$$G_u(u,\beta)u_\beta + G_\beta(u,\beta) = 0 \tag{6.28}$$

and the discretized sensitivity equations will have the form:

$$\int_\Omega (F_u^h(u^h,\beta)(u_\beta)^h + F_\beta^h(u^h,\beta))\ w_i\ d\Omega = 0 \tag{6.29}$$

$$G_u^h(u^h,\beta)(u_\beta)^h + G_\beta^h(u^h,\beta) = 0 \tag{6.30}$$

where $w_i$ is a "typical" basis function. The discretized state equations have the form:

$$\int_\Omega F^h(u^h,\beta)\ w_i\ d\Omega = 0 \tag{6.31}$$

$$G^h(u^h,\beta) = 0 \tag{6.32}$$

and the discrete sensitivity equations may be computed by carrying out the operations:

$$\frac{D}{D\beta}\int_\Omega F^h(u^h,\beta)\ w_i\ d\Omega = 0 \tag{6.33}$$

$$\frac{D}{D\beta}G^h(u^h,\beta) = 0 \tag{6.34}$$

Now because the domain of integration does not depend on $\beta$, we may bring the differentiation under the integral sign. Because the basis function $w_i$ does not depend in any way on $\beta$, we have only to compute the total derivative of $F^h$, so that the discrete sensitivity equations are:

$$\int_\Omega (F_u^h(u^h,\beta)u_\beta^h + F_\beta^h(u^h,\beta))\ w_i\ d\Omega = 0 \tag{6.35}$$

$$G_u^h(u^h,\beta)u_\beta^h + G_\beta^h(u^h,\beta) = 0 \tag{6.36}$$

But this system is identical in form to the discretized sensitivity equation, and hence their solutions must also be identical. □

## 6.5 Exact Example: Poiseuille Sensitivities

For the Poiseuille flow problem, consider the influence of $\lambda$, the strength of the inflow field, upon the solution defined by Equations (3.6)-(3.8). The continuous sensitivities may be immediately written down from the explicit formula for the solution:

$$u_\lambda = \frac{\partial u(x,y)}{\partial \lambda} = y(ymax - y) \tag{6.37}$$

$$v_\lambda = \frac{\partial v(x,y)}{\partial \lambda} = 0 \tag{6.38}$$

$$p_\lambda = \frac{\partial p(x,y)}{\partial \lambda} = 2(xmax - x)/Re \tag{6.39}$$

We can make some simple conclusions from this information. For instance, $\lambda$ has no influence on the vertical velocity anywhere. That's obvious, actually, since, for every value of $\lambda$ the vertical velocity is zero everywhere. Note also how the influence of $\lambda$ on the horizontal velocity varies: it is zero along the walls (where $u$ is always zero) and is greatest at the center line $y = ymax/2$, because changes in the velocity are proportional to the value of the velocity, and the velocity is greatest along the center line.

The Reynolds number $Re$ might also be considered a parameter for this problem, but would make a singularly uninteresting flow:

$$u_{Re} = 0 \tag{6.40}$$

$$v_{Re} = 0 \tag{6.41}$$

$$p_{Re} = -2\lambda(xmax - x)/Re^2 \tag{6.42}$$

The continuous sensitivity equations derived earlier may be checked by plugging in these continuous sensitivity functions. For the $Re$ parameter, note that the vertical momentum and continuity equations are identically zero, while the horizontal momentum equation reduces

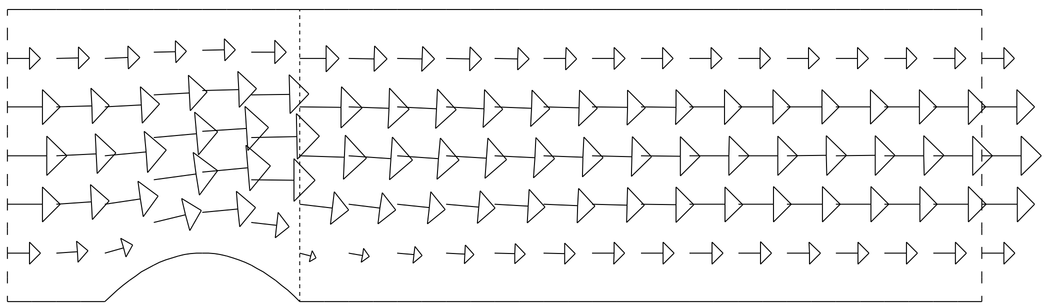

Figure 6.1: Discrete velocity sensitivity with respect to the inflow parameter $\lambda$.

to:

$$Re \frac{\partial p_{Re}}{\partial x} = -\frac{\partial p}{\partial x}, \tag{6.43}$$

which is easily verified by plugging in the values of $p$ and $p_{Re}$.

For the $\lambda$ parameter, the system simplifies to

$$-\frac{\partial^2 u_\lambda}{\partial y^2} + Re \frac{\partial p_\lambda}{\partial x} = 0 \tag{6.44}$$

$$\frac{\partial u_\lambda}{\partial x} = 0 \tag{6.45}$$

which is again easy to verify.

## 6.6 Computational Example: Flow Past a Bump

In this section, we display plots of the discrete sensitivities of the velocity for a flow problem with parameters $(\lambda, \alpha, Re)$. The base solution was calculated at parameter values $(0.25, 0.25, 5.0)$. The underlying grid had a mesh parameter of $h = 0.25$.

For clarity, we have only shown the sensitivities at every other node. Also, the vectors have been resized so that the largest vector has a fixed size; thus, comparisons of the strength of the influence of different parameters is not possible. However, in any particular plot, the

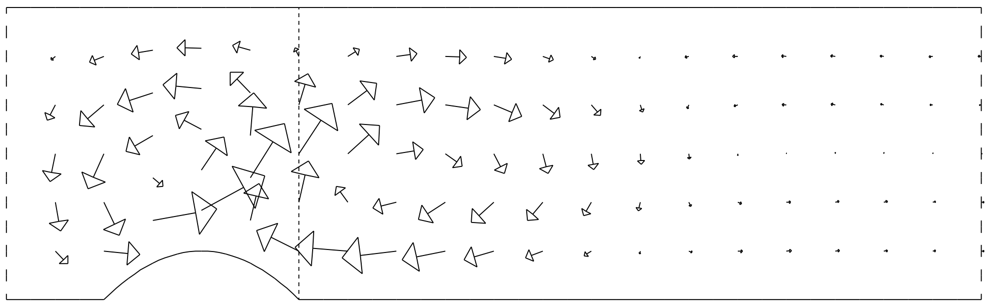

Figure 6.2: Discrete velocity sensitivity with respect to the *Re* parameter.

relative sizes of the vectors are meaningful, and represent the strength of the influence of the parameter on each particular velocity. A similar, but less enlightening, contour plot could be shown for pressure.

In the resulting plots, keep in mind that what we are displaying is a velocity *increment* that should be added to the current velocity if the given parameter is increased. The discrete velocity sensitivities for the inflow and *Re* parameters satisfy the discrete continuity equation, a fact suggested by the form of the displayed solutions in Figure 6.1 and Figure 6.2. Moreover, as can be judged from the plots, the $\lambda$-sensitivity equations have an inflow boundary term, while the *Re*-sensitivity equations have zero boundary conditions, but have source terms throughout the region. Note in Figure 6.1 how the inflow parameter affects the velocity field throughout the entire region, while Figure 6.2 shows that the influence of the *Re* parameter is restricted to the area near the bump.

## 6.7 Comparison of Sensitivities with Finite Difference Estimates

We have mentioned that the discrete sensitivities are an inexpensive estimate of the partial derivatives of the discrete state variables with respect to a parameter. We have also noted

Table 6.1: Finite difference check of $\lambda$-sensitivities.

| Variable | $\|u_\lambda^h\|_\infty$ | $\|\Delta(u^h)/\Delta(\lambda)\|_\infty$ | $\|\text{Difference}\|_\infty$ |
|---|---|---|---|
| U | 1.167 | 1.167 | 1.1E-08 |
| V | 0.2067 | 0.2067 | 6.7E-09 |
| P | 1.140 | 1.140 | 5.6E-08 |

Table 6.2: Finite difference check of $Re$-sensitivities.

| Variable | $\|u_{Re}^h\|_\infty$ | $\|\Delta(u^h)/\Delta(Re)\|_\infty$ | $\|\text{Difference}\|_\infty$ |
|---|---|---|---|
| U | 3.118E-03 | 3.118E-03 | 2.1E-09 |
| V | 2.460E-03 | 2.460E-03 | 1.3E-09 |
| P | 5.334E-02 | 5.334E-02 | 5.2E-08 |

that the accuracy of this approximation is limited, and depends on the fineness of the mesh parameter $h$.

To illustrate that the sensitivities can, in fact, provide a very good approximation in practice, Tables 6.1 and 6.2 compare the sensitivities with an estimate made using finite differences. The problem involves three parameters, $(\lambda, \alpha, Re)$, and our solution was computed at $(0.5, 0.5, 10.0)$, on a grid with $h = 0.125$. Here we will only look at the results for $Re$ and $\lambda$. These results show almost perfect agreement between the two calculations.

The formula for $\Delta(\lambda)$ requires the value of $\epsilon$, the relative accuracy for double precision arithmetic on the given computer, which, for the DEC Alpha, is roughly 1.1E-16:

$$\Delta(\lambda) = 10\epsilon^{\frac{1}{2}}\ sign(\lambda)\ (\|\lambda\| + 1), \tag{6.46}$$

which results in a relative perturbation of about 1.0E-07. The same sort of formula is used to compute $\Delta(Re)$.