# Chapter 3

# CONTINUOUS FLUID FLOW EQUATIONS

## 3.1 Introduction

In this chapter, we describe the *Navier Stokes equations.* These equations constrain, and generally determine, the behavior of the fluid flow in a given problem. In particular, they apply to the parameterized class of flow problems discussed in Chapter 2.

We examine the solution of a special example of our flow problems, called *Poiseuille flow*, which is useful in checking the correctness of the computational algorithms, and in understanding the behavior of flow solutions to problems with no bump, or with relatively low inflow velocity conditions.

We note that most flow problems are very difficult to solve. We then discuss *Newton's method*, which shows us how to turn a hard, nonlinear problem, into an easier sequence of linear problems.

## 3.2 The Navier Stokes Equations

The steady incompressible flow of a viscous fluid in a two dimensional region may be completely described by three functions of position: the horizontal velocity $u(x,y)$, vertical velocity $v(x,y)$, and pressure $p(x,y)$. The quantities $u$ and $v$ are really vector components of a single physical velocity which we may alternately write as $\mathbf{u}$.

We may refer to these functions as the *flow field solution.* They are an example of a set of *state variables*, that is, information which completely describes the state of some physical system.

Because they represent the behavior of a physical fluid, the functions $u$, $v$, and $p$ obey certain physical laws. Given our assumptions about the problem, we will find it appropriate to assume that these flow functions satisfy the Navier Stokes equations for stationary incompressible viscous flow at every point $(x,y)$ within the flow region $\Omega$.

A compact vector form of these equations may be written as:

$$\begin{aligned} -\Delta\mathbf{u} + Re(\mathbf{u}\cdot\nabla\mathbf{u} + \nabla p) &= 0 \qquad (3.1)\\ \nabla\cdot\mathbf{u} &= 0 \qquad (3.2)\end{aligned}$$

while we will prefer the verbose, but more readily grasped scalar version of the equations:

$$\begin{aligned} -(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + Re(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}) &= 0 \qquad (3.3)\\ -(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + Re(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y}) &= 0 \qquad (3.4)\\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \qquad (3.5)\end{aligned}$$

Because they constrain the behavior of state variables, these equations may be referred to as the *state equations.* The first pair of equations are sometimes referred to as the *equations* *of motion* or the *horizontal* and *vertical momentum equations.* The last equation is the *continuity* or *incompressibility equation.* The quantity $Re$ is the Reynolds number.

While the Navier Stokes equations constrain the behavior of the fluid within the flow region, a flow problem is only completely described when the Reynolds number and certain information about the flow along the boundary is given as well. In Chapter 2 we discussed the sort of boundary information that would be supplied for our problems, and we will assume that the boundary information and the value of $Re$ necessary to complete the problem have all been specified.

We have said that the Navier Stokes equations constrain any flow solution. That is, *if* there is a solution to the given problem, then it must satisfy the Navier Stokes equations. However, we are interested in knowing more about the behavior of possible solutions. Our questions include existence, uniqueness, and continuous dependence on data, or more precisely:

- Is there always at least one solution to a given problem?
- Is the solution to a given problem unique?
- Does the solution depend continuously, or even differentiably, on the boundary data and the Reynolds number?

For arbitrary boundary data, we cannot guarantee anything. Only if the boundary data satisfy certain "consistency" requirements, and if the region itself is bounded with a "reasonably smooth" boundary, can we guarantee the existence of at least one flow solution for any positive Reynolds number. It is easy to see that some constraints must be placed on boundary conditions. Otherwise, for instance, we could specify that fluid enter the region from all sides, an impossibility for incompressible flow.

If, furthermore, the Reynolds number is sufficiently small, then we can also guarantee that there is just one such flow solution.

Once the Reynolds number reaches a certain critical value, then uniqueness can no longer be guaranteed. If we consider the graph of the flow solution versus Reynolds number, then the graph may split or bifurcate into two or more distinct solution branches once a critical Reynolds value is reached. Such splitting can recur at certain higher Reynolds numbers as well.

However, below the first critical Reynolds value, and on any single solution branch, the flow solution will be continuously differentiable in terms of the Reynolds number.

For a summary of what is known about existence, uniqueness, and continuous dependence on the Reynolds number and boundary conditions, see Ladyzhenskaya [21] or Temam [25].

We will not delve further into these questions of existence and uniqueness. Instead, we will assume that when we pose one of our flow problems, there will be, at least locally, exactly one flow solution, which will depend smoothly on each of the problem parameters. We still face the daunting problem of *producing* that solution. To get an explicit formula for a solution to one of our problems requires the solution of a set of nonlinear partial differential equations. Even for the simple problems we will consider, there is no known method of producing such a solution. When faced with a general problem, then, we will find that we must turn to approximate methods in order to produce any useful information about a solution.

Before considering such methods, we turn to a special problem, for which an exact solution of the Navier Stokes equations is known.

## 3.3 Poiseuille Flow

Consider a horizontal channel (with no obstruction) of constant height *ymax*, and length *xmax*. Suppose that an incompressible, viscous fluid moves through the channel in accordance with the steady Navier Stokes equations. Suppose that the boundary conditions imposed on the flow are:

$$\begin{aligned} v(x,y) &= 0 \text{ along the boundary;} \\ u(0,y) &= \lambda y(ymax-y) \text{ along the inflow boundary;} \\ u(x,y) &= 0 \text{ along the upper and lower walls;} \\ \frac{\partial u}{\partial n}(xmax,y) &= 0 \text{ along the outflow boundary;} \\ p(xmax,ymax) &= 0. \end{aligned}$$

Then the Navier Stokes equations and boundary conditions are exactly satisfied by the following set of state variable functions:

$$u(x,y) = \lambda y(ymax-y) \tag{3.6}$$

$$v(x,y) = 0 \tag{3.7}$$

$$p(x,y) = 2\lambda(xmax-x)/Re. \tag{3.8}$$

This exact solution can be used as a very basic test of the computational correctness of our approximate Navier Stokes solver. The solution also makes it easy to compute the explicit form of the associated cost function, and gradients of the cost function with respect to the parameters $\lambda$ and $Re$, and to compare them to the results arrived at by computation. Finally, at least for small values of $Re$ and the inflow, we can introduce a small bump into the channel and still have the flow tend to return to the Poiseuille solution a short distance downstream from the bump.

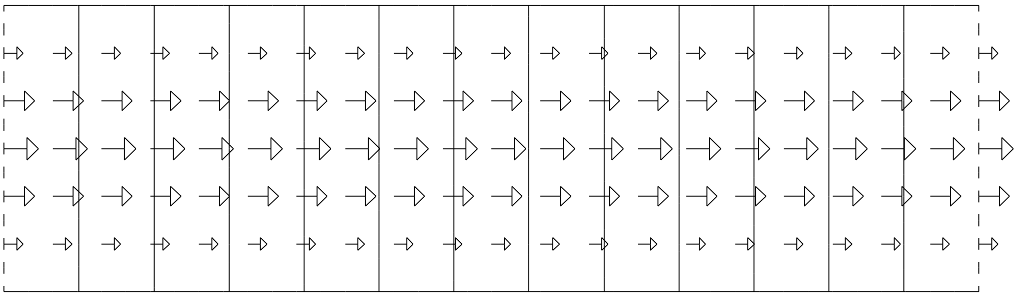

Figure 3.1: Velocity vectors and pressure contour lines for Poiseuille flow.

In some ways, the Poiseuille solution is not an ideal example. It's much too simple. It is very carefully constructed so that the nonlinear terms in the Navier Stokes equation exactly cancel out. This means that we will not see some of the nonlinear behavior that makes a general fluid flow problem so hard to solve.

## 3.4 Picard Iteration for the Continuous Navier Stokes Equations

Since there are no practical direct methods of solving the Navier Stokes equations over an arbitrary region, we need to consider methods of computing approximate solutions. While eventually we will turn to methods of discretization, we first consider several *iterative* schemes, which work directly with the original, continuous problem.

The primary appeal of these iterative schemes is that they replace a hard problem (solving a system of nonlinear partial differential equations) by a sequence of somewhat easier problems (solving a system of linear partial differential equations). While it is still beyond our abilities to solve even the linearized system, the ideas we discuss will help us to see how to handle similar problems when we finally move to the discretized case.

The first method we will discuss is known as *simple iteration* or *Picard iteration.* The idea is that we can try to solve the problem

$$F(X) = 0, \tag{3.9}$$

by "splitting" $F$ into a form such as:

$$F(X) = X - G(X), \tag{3.10}$$

or, in general, finding a "splitting" function $H(X, Y)$ so that:

$$F(X) = H(X, X). \tag{3.11}$$

Picard iteration would then consist of picking a starting point $X_0$ and solving the sequence of (generally implicit) problems:

$$H(X_{n+1}, X_n) = 0. \tag{3.12}$$

In many cases, if the decomposition that defines a Picard iteration is chosen properly, the iteration will converge for a wide range of starting guesses $X_0$, although convergence might be quite slow.

To apply Picard iteration to our flow problem, let us assume that we have a starting estimate $(u_0, v_0, p_0)$ of the solution, which can be any functions of $(x, y)$ defined over $\Omega$, and which need not satisfy the boundary conditions.

We now rewrite the Navier Stokes equations and the boundary conditions so that all terms appear on the left hand side only, so that formally we have reproduced the generic nonlinear system $F(X) = 0$. For our problem, this would result in the system of equations:

$$\begin{aligned} -(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + Re(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}) &= 0 \qquad (3.13) \\ -(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + Re(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y}) &= 0 \qquad (3.14) \end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 && (3.15)\\
v(x_\Gamma, y_\Gamma) &= 0 && (3.16)\\
u(0,y) - Inflow(y,\lambda) &= 0 && (3.17)\\
u(x_{wall}, y_{wall}) &= 0 && (3.18)\\
\frac{\partial u}{\partial n}(xmax, y) &= 0 && (3.19)\\
p(xmax, ymax) &= 0 && (3.20)
\end{aligned}$$

Here, the equation

$$v(x_\Gamma, y_\Gamma) = 0 \qquad (3.21)$$

is meant to assert that the vertical velocity $v$ is zero at every point $(x, y)$ that lies on the flow region boundary $\Gamma$. A similar symbolism is used to assert the conditions that apply just on the upper and lower walls.

Our decomposition of $F(X)$ into $H(X_{n+1}, X_n)$ will simply involve replacing each nonlinear term by a term that is linear in the "new" variables. Rather than subscripting our state variables, we will use the symbols $u_{old}$, $v_{old}$ and $p_{old}$ to refer to the data at the old point. Then our equations become:

$$\begin{aligned}
-(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + Re(u_{old}\frac{\partial u}{\partial x} + v_{old}\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}) &= 0 && (3.22)\\
-(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + Re(u_{old}\frac{\partial v}{\partial x} + v_{old}\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y}) &= 0 && (3.23)\\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 && (3.24)\\
v(x_\Gamma, y_\Gamma) &= 0 && (3.25)\\
u(0,y) - Inflow(y,\lambda) &= 0 && (3.26)\\
u(x_{wall}, y_{wall}) &= 0 && (3.27)\\
\frac{\partial u}{\partial n}(xmax, y) &= 0 && (3.28)\\
p(xmax, ymax) &= 0 && (3.29)
\end{aligned}$$

By linearizing four terms in our equations, we have come up with a linear system of partial differential equations, which are presumably easier to solve repeatedly than our original nonlinear system.

We will now discuss Newton's method for solving our nonlinear system. We will see that Newton's method converges faster, but only if the initial guess is close enough to the correct answer. This will mean that our best method will be a hybrid of Picard iteration, to move from an arbitrary starting point close to the correct solution, and Newton's method, which will rapidly finish the iteration once we are near enough.

## 3.5 Newton's Method for the Continuous Navier Stokes Equations

Newton's method for solving a system of nonlinear equations $F(X) = 0$ assumes that we have an approximate solution $X_0$ and a linearized operator $FP(X)$ which allows us to approximate the behavior of the function $F$ around any point $X$ as:

$$F(X) = F(X_0) + FP(X_0)\ (X - X_0) + O((X - X_0)^2). \tag{3.30}$$

If $F(X)$ is a single algebraic equation, then $FP(X)$ is simply the usual derivative function. If $F(X)$ is a system of algebraic equations, then $FP(X)$ is the Jacobian matrix. For our problem, however, $F(X)$ is not a function of a few real numbers, but of several real valued functions. The computation and use of the linearization operator $FP$ will be more complicated than in the discrete case, but the idea will still be the same.

Now if we drop the error term in Equation (3.30) and replace $F(X)$ by 0, we may solve for $X$, and arrive at the Newton estimate for the root:

$$X_1 = X_0 - FP^{-1}(X_0)\ F(X_0). \tag{3.31}$$

If our initial approximation is close to the correct answer $X^*$, and if the differential operator $FP$ is continuous and invertible at $X^*$, then we may expect that repeated application of this operation will produce a sequence of points $\{X_k\}$ which converge to $X^*$. We defer a detailed consideration of the convergence of Newton's method to the next chapter, when we discuss the discrete case that we will actually need to solve.

To apply Newton's method to our flow problem, let us assume that we have a starting estimate $(u_0, v_0, p_0)$ of the solution, which can be any functions of $(x, y)$ defined over $\Omega$, and which need not satisfy the boundary conditions. This estimate might, in fact, be the output of the Picard iteration.

We now rewrite the Navier Stokes equations and the boundary conditions so that all terms appear on the left hand side only, as in Equations (3.13) through (3.20), so that formally we have reproduced the generic Newton system $F(X) = 0$.

We may then symbolize the left hand side of these equations as $F(u, v, p)$ and our search for a solution of the original problem can be represented as seeking functions $(u, v, p)$ so that:

$$F(u, v, p) = 0. \tag{3.32}$$

Now we need to compute the linearization operator $FP$ associated with our nonlinear function. If our variables were simple algebraic quantities, then we would simply take the usual partial derivative $\frac{\partial}{\partial u}$. A term like $u^2$ would result in a derivative term of $2u$. However, our variables are functions and it is not immediately clear how to differentiate equations involving spatial derivatives of these functions, such as $u\frac{\partial u}{\partial x}$. A naive guess for the partial derivative might be the meaningless result $*\frac{\partial u}{\partial x} + u\frac{\partial *}{\partial x}$.

What's wrong with this result does suggest the proper approach, however. Where we placed a "*" in the above expression, we actually need something to operate on. Suppose we try to compute something like the *differential* instead of the derivative of our function? For an algebraic system, we would merely be computing a result like $2u\ du$, where the $du$ seems to add little. But for our situation, the equivalent of the $du$ will supply the missing operand.

Since the arguments of our functions aren't scalars, but are themselves functions, we must proceed carefully. In order to use a difference quotient approach, we will need to have a family of perturbations parameterized by a scalar $\epsilon$. So we first consider some arbitrary set of perturbation functions, which we will call $\tilde{u}$, $\tilde{v}$ and $\tilde{p}$. We will then evaluate the function $F$ at the family of "nearby" points $(u_0+\epsilon\tilde{u}, v_0+\epsilon\tilde{v}, p_0+\epsilon\tilde{p})$ and use a difference quotient to analyze the behavior of $F$ as $\epsilon \to 0$.

Our difference quotient will have the form:

$$FP(u_0,v_0,p_0)(\tilde{u},\tilde{v},\tilde{p}) = \lim_{\epsilon\to 0}\frac{F(u_0+\epsilon\tilde{u}, v_0+\epsilon\tilde{v}, p_0+\epsilon\tilde{p}) - F(u_0,v_0,p_0)}{\epsilon}. \tag{3.33}$$

If this limit exists, we will have found what is called the *Gateaux derivative* of $F$, which is a sort of directional derivative in abstract spaces. The derivation of the Gateaux derivative requires that we pick a particular direction or perturbation. If in fact the result is independent of the direction chosen, then we have actually found the *Frechet derivative.* The defining property of the Frechet derivative FP(x)(h) is that

$$\lim_{||h||\to 0}\frac{||F(x+h)-F(x)-FP(x)(h)||}{||h||} = 0. \tag{3.34}$$

If we carry out the limit operation, the resulting operator $FP(u_0,v_0,p_0)(\tilde{u},\tilde{v},\tilde{p})$ has the following form:

$$-(\frac{\partial^2\tilde{u}}{\partial x^2}+\frac{\partial^2\tilde{u}}{\partial y^2}) + Re\ (\tilde{u}\frac{\partial u_0}{\partial x} + u_0\frac{\partial\tilde{u}}{\partial x} + \tilde{v}\frac{\partial u_0}{\partial y} + v_0\frac{\partial\tilde{u}}{\partial y} + \frac{\partial\tilde{p}}{\partial x}) \tag{3.35}$$

$$-(\frac{\partial^2\tilde{v}}{\partial x^2}+\frac{\partial^2\tilde{v}}{\partial y^2}) + Re\ (\tilde{u}\frac{\partial v_0}{\partial x} + u_0\frac{\partial\tilde{v}}{\partial x} + \tilde{v}\frac{\partial v_0}{\partial y} + v_0\frac{\partial\tilde{v}}{\partial y} + \frac{\partial\tilde{p}}{\partial y}) \tag{3.36}$$

$$\frac{\partial\tilde{u}}{\partial x}+\frac{\partial\tilde{v}}{\partial y} \tag{3.37}$$

$$\tilde{v}(x_\Gamma, y_\Gamma) \tag{3.38}$$

$$\tilde{u}(0,y) \tag{3.39}$$

$$\tilde{u}(x_{wall}, y_{wall}) \tag{3.40}$$

$$\frac{\partial \tilde{u}}{\partial n}(xmax, y) \tag{3.41}$$

$$\tilde{p}(xmax, ymax) \tag{3.42}$$

Once we have the local linearization $FP$, we can write the desired estimate for the behavior of $F$:

$$\begin{aligned} & F(u_0 + \epsilon\tilde{u}, v_0 + \epsilon\tilde{v}, p_0 + \epsilon\tilde{p}) \\ & \quad = F(u_0, v_0, p_0) \quad + \quad \epsilon\ FP(u_0, v_0, p_0)\ (\tilde{u}, \tilde{v}, \tilde{p}) + O(\epsilon^2). \end{aligned} \tag{3.43}$$

Now we are ready to apply Newton's method to our problem. We evaluate the Navier Stokes function at our starting point, and because our starting point is not a root of $F$, we get a nonzero right hand side:

$$F(u_0, v_0, p_0) \neq 0. \tag{3.44}$$

How could we get an improved estimate of a root of $F$? Suppose we drop the $\epsilon^2$ term in the estimate of the behavior of $F$, assuming that its behavior can be entirely described by the linear part. Then, if there's a nearby point $(u_0 + \epsilon\tilde{u}, v_0 + \epsilon\tilde{v}, p_0 + \epsilon\tilde{p})$, at which $F$ is zero, we must have:

$$0 = F((u_0 + \epsilon\tilde{u}, v_0 + \epsilon\tilde{v}, p_0 + \epsilon\tilde{p}) \tag{3.45}$$

$$= F(u_0, v_0, p_0) + \epsilon\ FP(u_0, v_0, p_0)\ (\tilde{u}, \tilde{v}, \tilde{p}), \tag{3.46}$$

so, in other words, it must be true that

$$\epsilon FP(u_0, v_0, p_0)(\tilde{u}, \tilde{v}, \tilde{p}) = -F(u_0, v_0, p_0). \tag{3.47}$$

If we can solve this equation, then we produce an improved estimate of the root:

$$(u_1, v_1, p_1) = (u_0 + \epsilon\tilde{u}, v_0 + \epsilon\tilde{v}, p_0 + \epsilon\tilde{p}). \tag{3.48}$$

If we find that the function value at this new point is still unacceptably large, we may repeat the refinement process, linearizing around our current estimate and solving the resulting equations for the increments.

The heart of Newton's method is the equation for the increments. We should not be deluded by its simple form. For our case, this equation is actually a system of linear partial differential equations, which are by no means easy to solve. The important point to note here is that we know we can't solve the original problem, $F(u, v, p) = 0$, involving a system of *nonlinear* partial differential equations. Newton's method has offered us a way to get a good estimate of the solution to that problem, if we are willing to solve a series of related, *linear* problems.

The equations for the Newton increments to the Navier Stokes solution are related to the *Oseen equations* [22], which prescribe the behavior of a small perturbation flow $(u, v, p)$ that can be added to a pre-existing flow $(u_{old}, v_{old}, p_{old})$:

$$-(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) + Re\ (u\frac{\partial u_{old}}{\partial x} + u_{old}\frac{\partial u}{\partial x} + v\frac{\partial u_{old}}{\partial y} + v_{old}\frac{\partial u}{\partial y} + \frac{\partial p}{\partial x}) = 0 \qquad (3.49)$$

$$-(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) + Re\ (u\frac{\partial v_{old}}{\partial x} + u_{old}\frac{\partial v}{\partial x} + v\frac{\partial v_{old}}{\partial y} + v_{old}\frac{\partial v}{\partial y} + \frac{\partial p}{\partial y}) = 0 \qquad (3.50)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (3.51)$$

We will see the Oseen equations again, when we consider sensitivities.