Chapter 2

A PARAMETERIZED CLASS OF FLOW PROBLEMS

2.1 Introduction

In our research, we will look at a range of flow problems. These problems will have a great deal in common, and will differ only in a few small ways. In this chapter, we will define precisely how we generate the different problems, and how we can specify a few numbers that will completely describe any particular flow problem we are interested in.

This specification will consist of giving the number and values of a set of *flow parameters*, which will in turn control the Reynolds number, the inflow strength, and the shape of an obstructing bump. After describing how each of these parameters enters the problem, we finish by displaying a sample problem.

2.2 How the Parameters Enter the Problem

We begin by specifying the parts of our flow problem that will *never* change. Our flow region, which we will symbolize by Ω, will be a rectangular channel, with opposite corners at $(0,0)$ and $(xmax, ymax)$. A bump of an unspecified shape lies along the bottom of this channel, extending from $(1,0)$ to $(3,0)$. Over this range, the height of the bump is given by $y = Bump(x, \alpha)$, where the $Bump$ function has a simple form that depends on one or more parameters that we call α. We prefer to think that the $Bump$ function is always nonnegative, but this is not essential to our work. In fact, it will be more convenient to allow $Bump$ to take on negative values as well. In such a case, negative values of α would mean that our region would actually be enlarged, as the bottom of the channel is "hollowed out".

The boundary of our flow region, which we may symbolize by Γ, is simply the top, bottom, left and right extremities. The bottom extremity is not rectilinear, since it includes the surface of the bump. The left extremity may be called the inflow boundary, and the right extremity the outflow boundary.

The boundary conditions on u and v, the horizontal and vertical components of the velocity function, and on p, the pressure, will have the general form:

- $v(x, y) = 0$ on the boundary;
- $u(0, y) = Inflow(y, \lambda)$ where the $Inflow$ function depends on one or more parameters λ, as discussed in Section 2.4;
- $u(x, y) = 0$ on the upper and lower walls and the surface of the bump;
- $\frac{\partial u}{\partial n}(xmax, y) = 0$;
- $p(xmax, ymax) = 0$.

Finally, the fluid flowing in the channel has a positive kinematic viscosity ν which we have not yet specified. For convenience, instead of specifying ν, we will specify Re which we take to be $1/\nu$, and which, if the equations are appropriately nondimensionalized, we may call the Reynolds number. Specifying a particular positive value of Re is then equivalent to specifying ν.

We should note that a more common definition for the Reynolds number, at least among experimentalists, involves the ratio:

$$Re \equiv \frac{\rho v l}{\mu} \tag{2.1}$$

where ρ is a typical density, v a typical velocity, l a typical length, and μ the absolute viscosity. Our kinematic viscosity ν is equal to $\frac{\mu}{\rho}$. Since the velocity does not appear in our version of the Reynolds number, if we wish to translate our results in terms of the experimentalist Reynolds number, we must assume that a velocity on the order of 1 is a "typical velocity".

If we agree that all our flow problems will have these characteristics, then to completely specify a particular problem, we simply need to specify the value of Re, the number of parameters λ and α that we will use, the values of those parameters, and their meanings, that is, how the parameters are used to determine the $Bump$ and $Inflow$ functions.

Typically, when we wish to define a particular class of problems, we will specify a form for the $Bump$ and $Inflow$ functions, and the number of parameters to use for each; but we are not yet ready to specify fixed numerical values of λ, α and Re.

For the moment, we will assume that our choice of values for these parameters is *unconstrained*; that is, any set of parameter values is worth considering, in the sense that it corresponds to a physically meaningful problem. Generally, this sweeping assumption is not strictly true. We won't want negative Reynolds numbers, or bumps that protrude through the top of the region, or inflow functions that actually flow out! However, we will proceed as though these undesirable situations are unlikely to crop up in any of our problems.

As long as a choice of parameter values represents a legitimate problem, that is, a problem for which a physically sensible flow solution can be computed, then we call the values a *feasible set of parameter values* (meaning that it results in a meaningful flow) and we call the corresponding flow solution a *feasible flow solution* (meaning that it is the result of a particular choice of parameter values).

By using parameters, we have (partially) discretized the specification of the problem. Instead of considering absolutely arbitrary bump and inflow shapes, we only consider various numbers as possible values for the bump and inflow parameters. Instead of ranging over function spaces, we range over R^n. This makes it easy to define a particular problem, and search among a family of related problems.

We will now briefly discuss the meaning of the three different sets of parameters.

2.3 The Reynolds Number

The Reynolds number Re will appear in the Navier Stokes equations, which determine the fluid flow in our region. These equations will not be discussed until Chapter 3, but we wish to comment here on the role that the Reynolds number plays as a parameter.

The Navier Stokes equations involve a balance between "diffusion" and "convection" terms. The diffusion terms are linear and, by themselves, would tend to produce a simple flow. The convection terms are nonlinear, harder to solve, and introduce complications into the flow. The Reynolds number multiplies the convection terms, and hence controls the relative strength of this disruptive nonlinear influence.

In a diffusion-dominated flow, disturbances tend to die out, dampened by the fluid viscosity. Imagine, for instance, how the behavior of ripples in a pond would change if we replaced the water by cold maple syrup. Such a "thick" fluid tends to move in a very smooth and stable way, changing only slightly in response to small changes in local conditions, the boundary conditions, or the shape of the region.

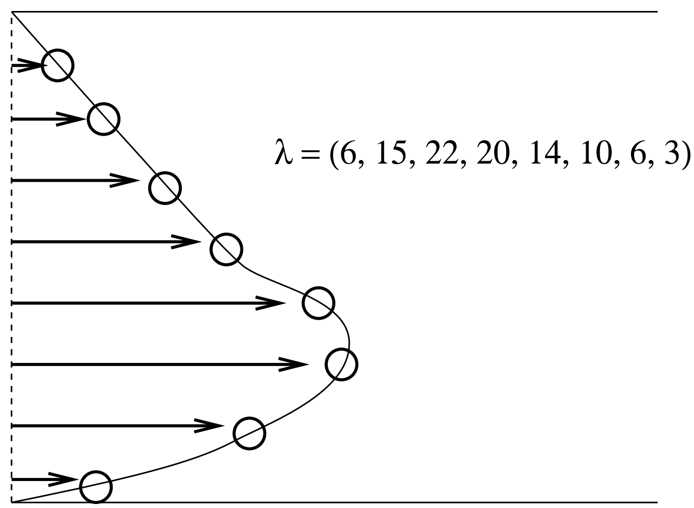

Figure 2.1: An inflow specified with eight parameters.

As Re increases, the convection terms dominate the flow. Now the flow field may change greatly over a small spatial region. The flow field solution is much more sensitive to changes in the problem conditions. Because of these facts, high Reynolds number flow problems are intrinsically harder to solve accurately.

We will not be solving the Navier Stokes equations directly, but the discretized version that we will end up working with inherits many of these characteristics. In particular, we will find that solutions to the discretized problem get very hard to compute for large values of Re. In some cases, we will only be able to find such a solution by solving a series of problems at lower values of Re, building up to a solution at the desired value of Re.

2.4 Specifying the Inflow Function

The next item to consider is the $Inflow$ function, which specifies the boundary condition to be applied to the horizontal velocity u along the left opening of the channel. While, in a real

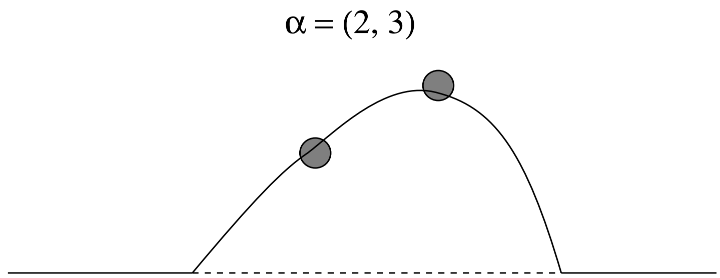

Figure 2.2: A bump specified with two parameters.

situation, there might be almost any sort of inflow data, we will want to set up a scheme where just a few numbers, represented by λ, are enough to specify the value of the inflow data at every point along the opening.

A natural way to do this is to let the values of λ represent the value of the inflow data at regularly spaced points along the opening, and then "fill in" the behavior of the function at other points in some simple way. While a wide variety of polynomials or piecewise polynomials may be used for this purpose, we will generally use *cubic splines* [9].

In Figure 2.1 we give an example of an inflow function which has been constructed by specifying eight desired values, which are listed in the figure. The intermediate behavior of the function is entirely determined by this data, and by the condition that the function be zero at both endpoints.

2.5 Specifying the Bump Shape

It remains for us to show how the height of the bump is to be described, that is, the form of the function $y = Bump(x, \alpha)$. The fact that we require the bump height y to be an explicit function of x means we have ruled out bumps which "fold over" or otherwise violate the requirement that the bump height be parameterizable in terms of x.

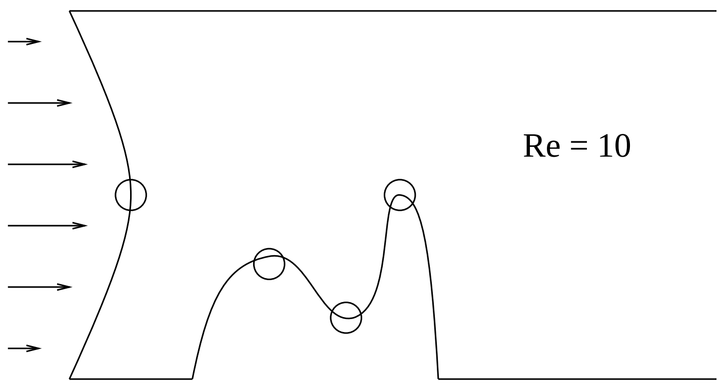

Figure 2.3: The flow problem $(\lambda, \alpha_1, \alpha_2, \alpha_3, Re) = (1.0, 1.0, 0.5, 1.5, 10.0)$.
The flow region actually extends further to the right.

We represent the bump height by setting it to zero at its endpoints, and specifying its value at regularly spaced points in between. We symbolize this set of values by α. We then construct the appropriate interpolating function so that we can produce values of the bump at any point over the interval defined by its endpoints. Usually we will take this interpolating function to be a cubic spline, although we may also choose piecewise linear or piecewise quadratic functions. In Figure 2.2 we show the generation of a typical bump shape, requiring the specification of two parameters, where interpolation by a cubic spline was used.

2.6 Example: A Five Parameter Problem

Now that we have detailed how any particular problem can be specified by giving the value of Re, and the number and values of the parameters α and λ, we will show a specific example.

The inflow boundary condition will be described by a single value $\lambda = 1.0$, the bump by the three values, $\alpha = (1.0, 0.5, 1.5)$, and the Reynolds number will be set to 10, so that the

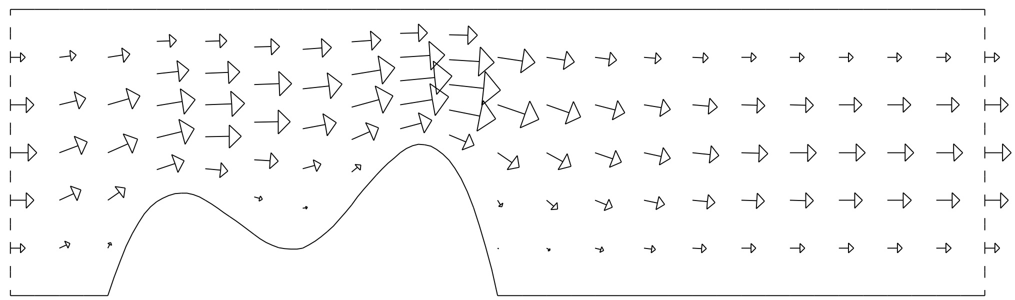

Figure 2.4: The velocity solution for the five parameter problem.

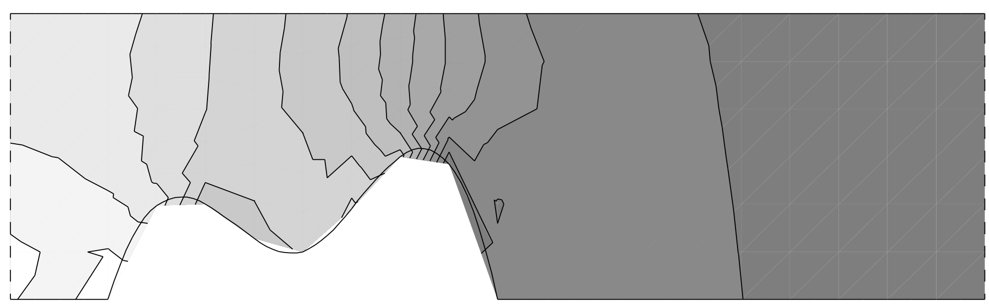

Figure 2.5: Contours of the pressure solution for the five parameter problem.

problem data can be summarized as:

$$(\lambda, \alpha_1, \alpha_2, \alpha_3, Re) = (1.0, 1.0, 0.5, 1.5, 10.0). \tag{2.2}$$

A schematic diagram of the corresponding problem is displayed in Figure 2.3.

Once we have defined the problem, the Navier Stokes equations, which we discuss shortly, will determine the behavior of the fluid at any point within the flow region. To emphasize this point, Figures 2.4 and 2.5 display the velocity and pressure solution fields of an approximate solution of the Navier Stokes equations, for the data we have just discussed. From the plots, the influence of the bump on the flow should be quite clear.

Now that we understand how the parameters will define the problem, it is time to turn to the question of how the problem can be solved, that is, how we can come up with values of the velocity and pressure that satisfy the boundary conditions and the physical laws that govern fluid behavior. To begin, we will look at the differential equations that describe the behavior of an incompressible viscous fluid.