

ON COLORINGS AVOIDING A RAINBOW CYCLE AND A FIXED MONOCHROMATIC SUBGRAPH

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ABSTRACT. Let H and G be two graphs on fixed number of vertices. An edge coloring of a complete graph is called (H, G) -good if there is no monochromatic copy of G and no rainbow (totally multicolored) copy of H in this coloring. As shown by Jamison and West [14], an (H, G) -good coloring of an arbitrarily large complete graph exists unless either G is a star or H is a forest. The largest number of colors in an (H, G) -good coloring of K_n is denoted $\max R(n, G, H)$. For graphs H which can not be vertex-partitioned into at most two induced forests, $\max R(n, G, H)$ has been determined asymptotically, [3]. Determining $\max R(n, G, H)$ is challenging for other graphs H , in particular for bipartite graphs or even for cycles. This manuscript treats the case when H is a cycle. The value of $\max R(n, G, C_k)$ is determined for all graphs G whose edges do not induce a star.

1. INTRODUCTION AND MAIN RESULTS

For two graphs G and H , an edge coloring of a complete graph is called (H, G) -good if there is no monochromatic copy of G and no rainbow (totally multicolored) copy of H in this coloring. The *mixed anti-Ramsey numbers*, $\max R(n; G, H)$, $\min R(n; G, H)$ are the maximum, minimum number of colors in an (H, G) -good coloring of K_n , respectively. The number $\max R(n; G, H)$ is closely related to the classical *anti-Ramsey number* $AR(n, H)$, the largest number of colors in an edge-coloring of K_n with no rainbow copy of H introduced by Erdős, Simonovits and Sós [9]. The number $\min R(n; G, H)$ is closely related to the classical multicolor Ramsey number $R_k(G)$, the largest n such that there is a coloring of edges of K_n with k colors and no monochromatic copy of G . The mixed Ramsey number $\min R(n; G, H)$ has been investigated in [3, 13, 11].

This manuscript addresses $\max R(n; G, H)$. As shown by Jamison and West [14], an (H, G) -good coloring of an arbitrarily large complete graph exists unless either G is a star or H is a forest. Let $a(H)$ be the smallest number of induced forests vertex-partitioning the graph H . This parameter is called a vertex arboricity. Axenovich and Iverson [3] proved the following.

Theorem 1. *Let G be a graph whose edges do not induce a star and H be a graph with $a(H) \geq 3$. Then $\max R(n; G, H) = \frac{n^2}{2} \left(1 - \frac{1}{a(H)-1}\right) (1 + o(1))$.*

When $a(H) = 2$, the problem is challenging and only few isolated results are known [3]. Even in the case when H is a cycle, the problem is nontrivial. This manuscript addresses this case. Since (C_k, G) -good colorings do not contain rainbow C_k , it follows that

$$(1) \quad \max R(n; G, C_k) \leq AR(n, C_k) = n \left(\frac{k-2}{2} + \frac{1}{k-1} \right) + O(1),$$

where the equality is proven by Montellano-Ballesteros and Neumann-Lara [16]. We show that $\max R(n; G, C_k) = AR(n; C_k)$ when G is either bipartite with large enough parts, or a graph with chromatic number at least 3. In case when G is bipartite with a “small” part, $\max R(n; G, C_k)$ depends mostly on G , namely, on the size of the “small” part. Below is the exact formulation of the main result.

If a graph G is bipartite, we let $s(G) = \min\{s : G \subseteq K_{s,r}, s \leq r \text{ for some } r\}$ and $t(G) = |V(G)| - s(G)$.

Theorem 2. *Let $k \geq 3$ be an integer and G be a graph whose edges do not induce a star. Let $s = s(G)$ and $t = t(G)$ if G is bipartite. There are constants $n_0 = n_0(G, k)$ and $g = g(G, k)$ such that for all $n \geq n_0$*

$$\max R(n; G, C_k) = \begin{cases} n\left(\frac{k-2}{2} + \frac{1}{k-1}\right) + O(1), & \text{if } (\chi(G) = 2 \text{ and } s \geq k) \text{ or } (\chi(G) \geq 3) \\ n\left(\frac{s-2}{2} + \frac{1}{s-1}\right) + g, & \text{otherwise} \end{cases}$$

Here $g = g(G, k) = ER^2(s+t, 3(s-1)k+t+1, k)$, where the number ER denotes the Erdős-Rado number stated in section 2. Note that it is sufficient to take $g(G, k) = 2^{c\ell^2 \log \ell}$, where $\ell = 3(s-1)k+t+1$.

We give the definitions and some observations in section 2, the proof of the main theorem in section 3 and some more accurate bounds for the case when $H = C_4$ in the last section of the manuscript.

2. DEFINITIONS AND PRELIMINARY RESULTS

First we shall define a few special edge colorings of a complete graph: lexical, weakly lexical, k -anticyclic, c^* and c^{**} .

Let $c : E(K_n) \rightarrow \mathbb{N}$ be an edge coloring of a complete graph on n vertices for some fixed n .

We say that c is a *weakly lexical* coloring if the vertices can be ordered v_1, \dots, v_n , and the colors can be renamed such that there is a function $\lambda : V(K_n) \rightarrow \mathbb{N}$, and $c(v_i v_j) = \lambda(v_{\min\{i, j\}})$, for $1 \leq i, j \leq n$. In particular, if λ is one to one, then c is called a *lexical* coloring.

We say that c is a *k -anticyclic* coloring if there is no rainbow copy of C_k , and there is a partition of $V(K_n)$ into sets V_0, V_1, \dots, V_m with $0 \leq |V_0| < k-1$ and $|V_1| = \dots = |V_m| = k-1$, where $m = \lfloor \frac{n}{k-1} \rfloor$, such that for i, j with $0 \leq i < j \leq m$, all edges between V_i and V_j have the same color, and the edges spanned by each $V_i, i = 0, \dots, m$ have new distinct colors using pairwise disjoint sets of colors.

We denote a fixed coloring from the set of k -anticyclic colorings of K_n such that the color of any edges between V_i and V_j is $\min\{i, j\}$ by c^* .

Finally, we need one more coloring, c^{**} , of K_n . Let c^{**} be a fixed coloring from the set of the following colorings of $E(K_n)$; let the vertex set $V(K_n)$ be a disjoint union of V_0, V_1, \dots, V_m with $0 \leq |V_0| < s-1$, $|V_1| = \dots = |V_{m-1}| = s-1$, and $|V_m| = k-1$, where $m-1 = \lfloor \frac{n-k+1}{s-1} \rfloor$. Let the color of each edge between V_i and V_j for $0 \leq i < j \leq m$ be i . Color the edges spanned by each $V_i, i = 0, \dots, m$ with new distinct colors using pairwise disjoint sets of colors.

For a coloring c , let the number of colors used by c be denoted by $|c|$. Observe that c^* is a blow-up of a lexical coloring with parts inducing rainbow complete subgraphs. Any monochromatic bipartite subgraph in c^* and c^{**} is a subgraph of $K_{k-1, t}$ and $K_{s-1, t}$ for some t , respectively. Also we easily see that if c is k -anticyclic, then

$$(2) \quad |c| \leq |c^*| = n \left(\frac{k-2}{2} + \frac{1}{k-1} \right) + O(1),$$

$$(3) \quad |c^{**}| = n \left(\frac{s-2}{2} + \frac{1}{s-1} \right) + O(1).$$

Let $K = K_n$. For disjoint sets $X, Y \subseteq V$, let $K[X]$ be the subgraph of K induced by X , and let $K[X, Y]$ be the bipartite subgraph of K induced by X and Y . Let $c(X)$ and $c(X, Y)$ denote the sets of colors used in $K[X]$ and $K[X, Y]$, respectively by a coloring c .

Next, we state a canonical Ramsey theorem which is essential for our proofs.

Theorem 3 (Deuber [7], Erdős-Rado [8]). *For any integers m, l, r , there is a smallest integer $n = ER(m, l, r)$, such that any edge-coloring of K_n contains either a monochromatic copy of K_m , a lexically colored copy of K_l , or a rainbow copy of K_r .*

The number ER is typically referred to as Erdős-Rado number, with best bound in the symmetric case provided by Lefmann and Rödl [15], in the following form: $2^{c_1 \ell^2} \leq ER(\ell, \ell, \ell) \leq 2^{c_2 \ell^2 \log \ell}$, for some constants c_1, c_2 .

3. PROOF OF THEOREM 2

If G is a graph with chromatic number at least 3, then $\max R(n; G, C_k) = n \left(\frac{k-2}{2} + \frac{1}{k-1} \right) + O(1)$ as was proven in [3].

For the rest of the proof we shall assume that G is a bipartite graph with $s = s(G)$, $t = t(G)$, and $G \subseteq K_{s,t}$. Let $K = K_n$. If $s \geq k$, then the lower bound on $\max R(n; G, C_k)$ is given by c^* , a special k -anticyclic coloring. The upper bound follows from (1).

Suppose $s < k$. The lower bound is provided by a coloring c^{**} . Since $\max R(n; G, C_k) \leq \max R(n; K_{s,t}, C_k)$, in order to provide an upper bound on $\max R(n; G, C_k)$, we shall be giving an upper bound on $\max R(n; K_{s,t}, C_k)$.

The idea of the proof is as follows. We consider an edge coloring c of $K = (V, E)$ with no monochromatic $K_{s,t}$ and no rainbow C_k , and estimate the number of colors in this coloring by analyzing specific vertex subsets: L, B, C , where L is the vertex set of the largest weakly lexically colored complete subgraph, B is the set of vertices in $V \setminus L$ which “disagrees” with coloring of L on some edges incident to the initial part of L , and C is the set of vertices in $V \setminus L$ which “disagrees” with coloring of L on some edges incident to the terminal part of L . Let $V' = V \setminus L$. We are counting the colors in the following order: first colors induced by V' which are not used on any edges incident to L or any edges induced by L , then colors used on edges between V' and L which are not induced by L , finally colors induced by L .

Now, we provide a formal proof. Let $\tau := 2(s-1)k + t$. Assume that n is sufficiently large such that $n \geq ER(s+t, \tau + (s-1)k + 1, k)$. Let c be a coloring of $E(K)$ with no monochromatic copy of $K_{s,t}$ and no rainbow copy of C_k , $c : E(K) \rightarrow \mathbb{N}$. Then there is a lexically colored copy of $K_{\tau+(s-1)k+1}$ by the canonical Ramsey theorem. Let L be a vertex set of a largest weakly lexically colored K_q , $q \geq \tau + (s-1)k + 1$, say $L = \{x_1, \dots, x_q\}$ and $c(x_i x_j) = \lambda(x_i)$ for $1 \leq i < j \leq q$, for some function $\lambda : L \rightarrow \mathbb{N}$. If $X = \{x_{i_1}, \dots, x_{i_\ell}\} \subseteq L$ and $\lambda(x_{i_1}) = \dots = \lambda(x_{i_\ell}) = j$ for some j , then we denote $\lambda(X) = j$. We write, for $i \leq j$, $x_i L x_j := \{x_i, x_{i+1}, \dots, x_j\}$, and for $i > j$, $x_i L x_j := \{x_i, x_{i-1}, \dots, x_j\}$. We say that x_i precedes x_j if $i < j$. Let T, T', T'' , and T''' be the tails of L of size $t, \tau - (s-1)k, \tau$, and $\tau + (s-1)k$ respectively, i.e., $T := \{x_{q-t+1}, x_{q-t+2}, \dots, x_q\}$, $T' := \{x_{q-\tau+(s-1)k+1}, x_{q-\tau+(s-1)k+2}, \dots, x_q\}$,

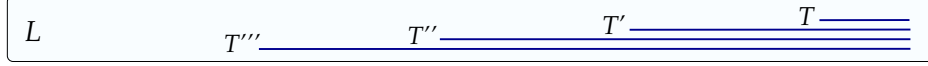


FIGURE 1. T , T' , T'' , and T'''

$T'' := \{x_{q-\tau+1}, x_{q-\tau+2}, \dots, x_q\}$, and $T''' := \{x_{q-\tau-(s-1)k+1}, x_{q-\tau-(s-1)k+2}, \dots, x_q\}$, see Figure 1.

Let $V' = V \setminus L = A \cup B \cup C$, where

$$\begin{aligned} A &:= \{v \in V' \mid c(vx) = \lambda(x), x \in L \setminus \{x_q\}\}, \\ B &:= \{v \in V' \mid \text{there exists } y \in L \setminus T'' \text{ such that } c(vy) \neq \lambda(y)\}, \\ C &:= \{v \in V' \mid c(vx) = \lambda(x), x \in L \setminus T'', \\ &\quad \text{and there exists } y \in T'' \setminus \{x_q\} \text{ such that } c(vy) \neq \lambda(y)\}. \end{aligned}$$

Note that $A = \emptyset$ since otherwise L is not the largest weakly colored complete subgraph. Thus

$$V = L \cup B \cup C.$$

Let $C_0 := c(L) \cup c(V', L)$. In the first part of the proof we bound $|(c(C) \cup c(C, B)) \setminus C_0| + |c(C, L) \setminus c(L)|$, in the second part we bound $|c(B) \setminus C_0| + |c(B, L) \setminus c(L)| + |c(L)|$.

Claim 0.1 Let $x \in L \setminus T$. Then $|\{y \in L \setminus T \mid \lambda(x) = \lambda(y)\}| \leq s - 1$.

If this claim does not hold, the corresponding y 's and T induce a monochromatic $K_{s,t}$.

Claim 0.2 Let $y, y' \in L \setminus T$ such that $|yLy'| > (s - 1)\ell + 1$ for some $\ell \geq 0$. Then $|c(yLy')| \geq \ell + 1$.

It follows from Claim 0.1.

Claim 0.3 Let $v, v' \in V'$ and $y, y' \in L \setminus T$. Let P be a rainbow path from v to v' in V' with $1 \leq |V(P)| \leq k - 2$ and colors not from C_0 . If $c(vy) \neq \lambda(y)$, $c(v'y') \notin \{c(vy), \lambda(y)\}$, and $|yLy'| > (s - 1)(k - |V(P)|) + 1$, then there is a rainbow C_k induced by $V(P) \cup yLy'$.

Indeed, by Claim 0.2, $|c(yLy')| \geq k - |V(P)| + 1$. Hence $|c(yLy') \setminus \{c(vy), c(v'y')\}| \geq k - |V(P)| - 1$. So we can find a rainbow path on $k - |V(P)|$ vertices in L with endpoints y and y' of colors from $c(yLy') \setminus \{c(vy), c(v'y')\}$, which together with $V(P)$ induce a rainbow C_k since colors of P are not from C_0 .

PART 1

We shall show that $|(c(C) \cup c(C, B)) \setminus C_0| + |c(C, L) \setminus c(L)| \leq \text{const}$.

Claim 1.1 $|C| \leq ER(s + t, \tau + 1, k)$.

Suppose $|C| > ER(s + t, \tau + 1, k)$. Then there is a lexically colored copy of a complete subgraph on a vertex set $Y \subseteq C$ of size $\tau + 1$. Then $(L \cup Y) \setminus T''$ is weakly lexical, which contradicts the maximality of L .

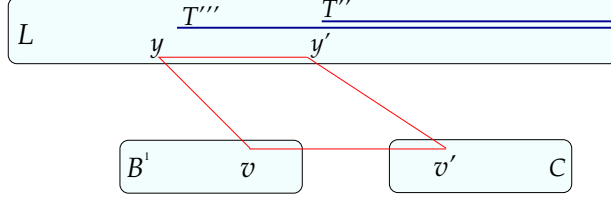


FIGURE 2. A rainbow C_k in Claim 1.3

Claim 1.2 $|c(C, L) \setminus c(L)| \leq \tau|C|$.

$|c(C, L) \setminus c(L)| \leq |c(C, T''')| \leq \tau|C|$ by the definition of C .

Claim 1.3 $\left| (c(C) \cup c(C, B)) \setminus C_0 \right| \leq \binom{ER(s+t, \tau+(s-1)k+1, k)}{2}$.

Let $B = B^1 \cup B^2$, where $B^1 := \{v \in B \mid \text{there exists } y \in L \setminus T''' \text{ with } c(vy) \neq \lambda(y)\}$, and $B^2 := B \setminus B^1$.

First, we show that $c(C, B^1) \subseteq C_0$. Assume that $c(v'v) \notin C_0$ for some $v \in B^1$ and $v' \in C$ with $c(vy) \neq \lambda(y)$ for some $y \in L \setminus T'''$ and $c(v'x) = \lambda(x)$ for any $x \in L \setminus T''$. From Claim 0.1, we can find $y' \in \{x_{q-\tau-2(s-1)+1}, x_{q-\tau-2(s-1)+1}, \dots, x_{q-\tau}\} \subseteq T''' \setminus T''$ such that $\lambda(y')$ is neither $c(vy)$ nor $\lambda(y)$. Since $\lambda(y') = c(v'y')$, we have that $c(v'y') \notin \{c(vy), \lambda(y)\}$. Moreover we have $|yLy'| > (s-1)(k-2)+1$. By Claim 0.3, there is a rainbow C_k induced by $\{v, v'\} \cup yLy'$, see Figure 2.

Second, we shall observe that $|B^2 \cup C| \leq ER(s+t, \tau+(s-1)k+1, k)$ by the argument similar to one used in Claim 1.1. We see that otherwise $B^2 \cup C$ contains a lexically colored complete subgraph on $\tau+(s-1)k+1$ vertices, which together with $L - T'''$ gives a larger than L weakly lexically colored complete subgraph.

PART 2

We shall show that $|c(B) \setminus C_0| + |c(B, L) \setminus c(L)| + |c(L)| \leq n \left(\frac{s-2}{2} + \frac{1}{s-1} \right)$.

In order to count the number of colors in B and (B, L) , we consider a representing graph of these colors as follows. First, consider a set E' of edges from $K[B]$ having exactly one edge of each color from $c(B) \setminus C_0$. Second, consider a set of edges E'' from the bipartite graph $K[B, L]$ having exactly one edge of each color from $c(B, L) \setminus c(L)$. Let G be a graph with edge-set $E' \cup E''$ spanning B . Then $|c(B) \setminus C_0| + |c(B, L) \setminus c(L)| = |E(G)|$.

We need to estimate the number of edges in G . Let B_1, \dots, B_p be sets of vertices of the connected components of $G[B]$. Let L_1, \dots, L_p be sets of the neighbors of B_1, \dots, B_p in L respectively, i.e., for $1 \leq i \leq p$, $L_i := \{x \in L \mid \{x, y\} \in E(G) \text{ for some } y \in B_i\}$. Let

$$G_1 := \bigcup_{i : |E(G[B_i, L_i])| \leq 1} G[B_i],$$

$$G_2 := \bigcup_{i : |E(G[B_i, L_i])| \geq 2} G[B_i \cup L_i].$$

Let G'_1, \dots, G'_{p_1} be the connected components of G_1 , and let G''_1, \dots, G''_{p_2} be the connected components of G_2 . See Figure 3 for an example of G_1 and G_2 .

Claim 2.1 We may assume that $V(G) \cap L \subseteq L \setminus T'$.

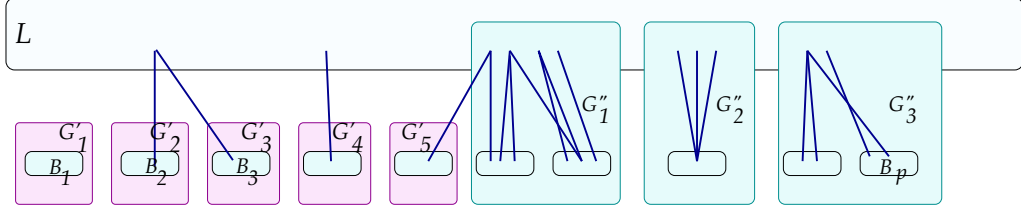


FIGURE 3. G_1 and G_2

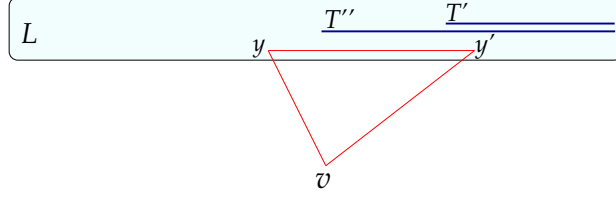


FIGURE 4. A rainbow C_k in Claim 2.1 and Claim 2.2-1.(1)

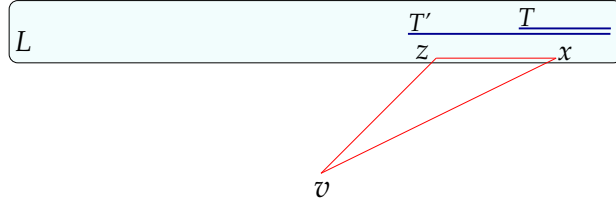


FIGURE 5. A rainbow C_k in Claim 2.2-1.(2)

For a fixed $v \in B$, let ω be a color in $c(v, L) \setminus c(L)$, if such exists. Let $L(\omega) := \{x \in L \mid c(vx) = \omega\}$. Suppose $L(\omega) \subseteq T'$. Since $v \in B$, there exists $y \in L \setminus T''$ such that $c(vy) \neq \lambda(y)$. Let $y' \in L(\omega) \subseteq T'$. Then $c(vy') \notin \{c(vy), \lambda(y)\}$. Since $|yLy'| > (s-1)k+1 > (s-1)(k-1)+1$, there is a rainbow C_k induced by $\{v\} \cup yLy'$ by Claim 0.3. Therefore $L(\omega) \cap (L \setminus T') \neq \emptyset$. Hence we can choose edges for the edge set E'' of G only from $K[B, L \setminus T']$.

Claim 2.2 For any i, j , $1 \leq i \leq p$, $1 \leq j \leq p_2$, we have that $K[B_i, T]$, $K[V(G''_j), T]$ are monochromatic complete bipartite subgraphs.

1. Fix i , $1 \leq i \leq p$. We show that $K[B_i, T]$ is monochromatic. Let $v \in B_i$ and $y \in L \setminus T''$ with $c(vy) \neq \lambda(y)$.

- (1) For any $y' \in T'$, $c(vy')$ is either $c(vy)$ or $\lambda(y)$. Indeed if $c(vx) \notin \{c(vy), \lambda(y)\}$, then there is a rainbow C_k induced by $\{v\} \cup yLy'$ by Claim 0.3, see Figure 4.
- (2) $|c(v, T)| = 1$. Indeed, let $L^y = \{x \in T' \setminus T \mid \lambda(x) \neq c(vy) \text{ and } \lambda(x) \neq \lambda(y)\}$. Then by Claim 0.1, $|L^y| \geq |T' \setminus T| - 2(s-1) + 1 > (s-1)(k-3) + 1$. Hence $|c(L^y)| \geq k-2$ by Claim 0.2. Let z be the vertex in L^y preceding every other vertex in L^y . Suppose there is $x \in T$ such that $c(vx) \neq c(vz)$. Since $c(L^y) \subseteq c(zLx)$, there exists a rainbow path from z to x on $k-1$ vertices in T' of colors disjoint from $\{c(vy), \lambda(y)\}$. So there is a rainbow C_k induced by $\{v\} \cup zLx$, see Figure 5. Therefore for any $x \in T$, $c(vx) = c(vz) \in \{c(vy), \lambda(y)\}$.

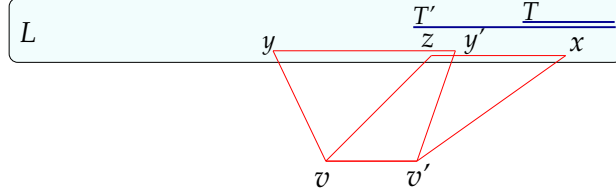


FIGURE 6. Rainbow C_k 's in Claim 2.2-1.(3)

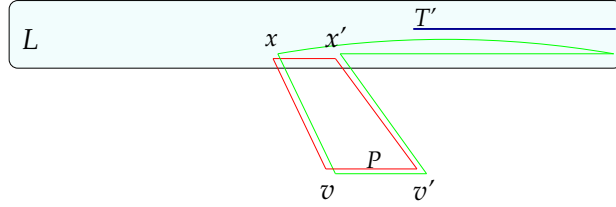


FIGURE 7. Rainbow C_k 's in Claim 2.2-2.(1): red when $|P| = k - 2$, green when $|P| < k - 2$.

- (3) For any neighbor v' of v in $G[B_i]$, if such exists, $c(v', T) = c(v, T)$. Indeed, we see that for any $y' \in T'$, $c(v'y') \in \{c(vy), \lambda(y)\}$, otherwise there is a rainbow C_k induced by $\{v, v'\} \cup yLy'$ by Claim 0.3. Also we see that for any $x \in T$, $c(v'x) = c(vz) \in \{c(vy), \lambda(y)\}$, where z is defined above; otherwise there is a rainbow C_k induced by $\{v, v'\} \cup zLx$, see Figure 6. Therefore $c(v', T) = c(v, T)$.
- (4) Since $G[B_i]$ is connected, $K[B_i, T]$ is monochromatic of color $c(vz)$.

Note that to avoid a monochromatic $K_{s,t}$, we must have that $|B_i| \leq s - 1 \leq k - 2$ for $1 \leq i \leq p$.

2. Fix j , $1 \leq j \leq p_2$. We show that $K[V(G''_j), T]$ is monochromatic.

- (1) $K[V(G''_j) \cap L, T]$ is monochromatic. Indeed, since G''_j , a connected component of G , is a union of $G[B_i \cup L_i]$'s satisfying $|E(G[B_i, L_i])| \geq 2$, by the connectivity, it is enough to show that $\lambda(x) = \lambda(x')$ for any $x, x' \in L_i$ for L_i in G''_j , where x precedes x' . From Claim 2.1, we may assume that x, x' are in $L \setminus T'$. Suppose $\lambda(x) \neq \lambda(x')$. Let $v, v' \in B_i$ such that $\{v, x\}$ and $\{v', x'\}$ are edges of G (possibly $v = v'$). Let P denote a set of vertices on a path from v to v' in $G[B_i]$. Then $1 \leq |P| \leq k - 2$ since $|B_i| \leq k - 2$. If $|P| = k - 2$, then $P \cup \{x, x'\}$ induces a rainbow C_k , otherwise so does $P \cup \{x\} \cup x'Lx_q$ from Claim 0.3, see Figure 7. Therefore $\lambda(x) = \lambda(x')$.
- (2) $K[V(G''_j), T]$ is monochromatic. To prove this, consider i such that $G[B_i, L_i] \subseteq G''_j$. Observe first that $K[B_i, T]$ and $K[L_i, T]$ are monochromatic by 1.(4) and 2.(1). Next, we shall show that $c(B_i, T) = \lambda(L_i)$. Suppose $c(B_i, T) \neq \lambda(L_i)$ for some i such that $G[B_i \cup L_i] \subseteq G''_j$. Let $v, v' \in B_i$ and $x, x' \in L_i$ such that $\{v, x\}$ and $\{v', x'\}$ are edges of G (possibly either $v = v'$ or $x = x'$). Since $|E(G[B_i, L_i])| \geq 2$, we can find such vertices. So $c(vx) \neq c(v'x')$ and $\{c(vx), c(v'x')\} \cap c(L) = \emptyset$. We may assume that $x, x' \in L \setminus T'$ by Claim 2.1. Since $c(B_i, T) \neq \lambda(L_i)$, $c(vx) = c(v'x') = c(B_i, T)$, otherwise there is a rainbow C_k induced by $\{v\} \cup xLx_q$ or $\{v'\} \cup x'Lx_q$ by Claim 0.3, see Figure 8. Then it contradicts the fact that $c(vx) \neq c(v'x')$.

We have that for any i such that $G[B_i, L_i] \subseteq G''_j$, $c(B_i, T) = \lambda(L_i)$. This implies that $K[B_i \cup L_i, T]$ is monochromatic of color $\lambda(L_i)$. Since G''_j is connected and B_i 's are disjoint, we have that for any i, i' such that $G[B_i, L_i], G[B_{i'}, L_{i'}] \subseteq G''_j$, $L_i \cap L_{i'} \neq \emptyset$, so $\lambda(L_i) = \lambda(L_{i'}) = \lambda$, for some λ . Therefore $K[V(G''_j), T]$ is monochromatic of color λ .

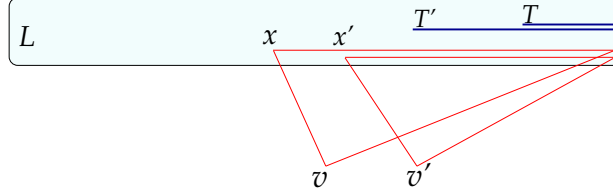


FIGURE 8. Rainbow C_k 's for Claim 2.2-2.(2).

Claim 2.3 $1 \leq |V(G'_i)|, |V(G''_j)| \leq s - 1$ for $1 \leq i \leq p_1$ and $1 \leq j \leq p_2$.
This claim now follows from the previous instantly.

The following claim deals with a small quadratic optimization problem we shall need.

Claim 2.4 Let $n, s \in \mathbb{N}$. Suppose n is sufficiently large and $s \geq 2$. Let $\xi_1, \dots, \xi_m \in \mathbb{N}$, $1 \leq \xi_i \leq s - 1$ and $\sum_{i=1}^m \xi_i \leq n$. Then

$$\sum_{i=1}^m \binom{\xi_i - 1}{2} \leq n \left(\frac{s-4}{2} + \frac{1}{s-1} \right).$$

The equality holds if and only if $m = \frac{n}{s-1}$ and $\xi_1 = \dots = \xi_m = s - 1$.
See the appendix A for the proof.

Claim 2.5 $|c(B) \setminus C_0| + |c(B, L) \setminus c(L)| + |c(L)| = |E(G)| + |c(L)| \leq n \left(\frac{s-2}{2} + \frac{1}{s-1} \right)$.

We have that

$$|E(G)| \leq (|E(G_1)| + p_1) + |E(G_2)| = \sum_{i=1}^{p_1} |E(G'_i)| + p_1 + \sum_{i=1}^{p_2} |E(G''_i)|.$$

Moreover each component G''_i of G_2 contributes at most 1 to $|c(L)|$ by Claim 2.2, and G_1 and G_2 are vertex disjoint. So

$$|c(L)| \leq n - |V(G_1)| - |V(G_2)| + p_2 = n - \sum_{i=1}^{p_1} |V(G'_i)| - \sum_{i=1}^{p_2} |V(G''_i)| + p_2$$

Hence we have

$$\begin{aligned} & |c(B) \setminus C_0| + |c(B, L) \setminus c(L)| + |c(L)| = |E(G)| + |c(L)| \\ & \leq \sum_{i=1}^{p_1} |E(G'_i)| + p_1 + \sum_{i=1}^{p_2} |E(G''_i)| + n - \sum_{i=1}^{p_1} |V(G'_i)| - \sum_{i=1}^{p_2} |V(G''_i)| + p_2 \\ & = \sum_{i=1}^{p_1} |E(G'_i)| + \sum_{i=1}^{p_2} |E(G''_i)| - \sum_{i=1}^{p_1} (|V(G'_i)| - 1) - \sum_{i=1}^{p_2} (|V(G''_i)| - 1) + n \\ & \leq \sum_{i=1}^{p_1} \binom{|V(G'_i)|}{2} + \sum_{i=1}^{p_2} \binom{|V(G''_i)|}{2} - \sum_{i=1}^{p_1} (|V(G'_i)| - 1) - \sum_{i=1}^{p_2} (|V(G''_i)| - 1) + n \\ & = \sum_{i=1}^{p_1} \binom{|V(G'_i)| - 1}{2} + \sum_{i=1}^{p_2} \binom{|V(G''_i)| - 1}{2} + n \end{aligned}$$

For $1 \leq i \leq p_1 + p_2$, let

$$\xi_i = \begin{cases} |V(G'_i)|, & \text{if } 1 \leq i \leq p_1 \\ |V(G''_{i-p_1})|, & \text{if } p_1 + 1 \leq i \leq p_1 + p_2 \end{cases}.$$

Then $\sum_{i=1}^{p_1+p_2} \xi_i \leq n$ and $1 \leq \xi_i \leq s-1$ for $1 \leq i \leq p_1 + p_2$ by Claim 2.3. From Claim 2.4, we get

$$\begin{aligned} & |c(B) \setminus C_0| + |c(B, L) \setminus c(L)| + |c(L)| \\ & \leq \sum_{i=1}^{p_1+p_2} \binom{\xi_i - 1}{2} + n \leq n \left(\frac{s-2}{2} + \frac{1}{s-1} \right). \end{aligned}$$

This concludes Part 2 of the proof.

Combining Parts 1 and 2, we see that the total number of colors is at most

$$\begin{aligned} & \left| (c(C) \cup c(C, B)) \setminus C_0 \right| + |c(C, L) \setminus c(L)| + |c(B) \setminus C_0| + |c(B, L) \setminus c(L)| + |c(L)| \\ & \leq \binom{ER(s+t, \tau + (s-1)k+1, k)}{2} + \tau ER(s+t, \tau+1, k) + n \left(\frac{s-2}{2} + \frac{1}{s-1} \right) \\ & \leq g + n \left(\frac{s-2}{2} + \frac{1}{s-1} \right), \end{aligned}$$

where $g = g(s, t, k) = ER^2(s+t, \tau + (s-1)k+1, k)$.

4. MORE PRECISE RESULTS FOR C_4

For a coloring c of $E(K_n)$ and a vertex v , let $N_c(v)$ be the set of colors between v and $V(K_n) \setminus \{v\}$, not used on edges spanned by $V(K_n) \setminus \{v\}$. Let $n_c(v) = |N_c(v)|$. Note that $c(uv) \in N_c(u) \cap N_c(v)$ if and only if the color $c(uv)$ is used only on the edge uv in the coloring c . We call this color a *unique color* in c . For a path $P = v_1 v_2 \cdots v_k$, we say that the path P is *good* if $c(v_i v_{i+1}) \in N_c(v_i)$ for $i = 1, \dots, k-1$.

Lemma 4. *Let c be an edge-coloring of K_n with no rainbow C_k . If for all $v \in V(K_n)$, $n_c(v) \geq k-2$, then $(k-1) \mid n$ and c is k -anticyclic.*

Proof. Let c be an edge-coloring of K_n with no rainbow C_k . Suppose for all $v \in V(K_n)$, $n_c(v) \geq k-2$. Then for any $v \in V$, we can find a good path of length $k-2$ starting at v by a greedy algorithm. Let this path be $v_1 v_2 \cdots v_{k-1}$, and let $c(v_i v_{i+1}) = i$ for $i = 1, \dots, k-2$. Let $V_0 = \{v_1, \dots, v_{k-1}\}$.

Claim 1 For any $u \in V \setminus V_0$, $c(uv_1) = 1$ or $c(uv_1) \notin N_c(v_1)$.

Assume that $c(uv_1) \in N_c(v_1)$. If $c(uv_1) \neq 1$ then $c(uv_{k-1})$ must be the same as $c(uv_1)$, otherwise $v_1 \cdots v_{k-1} uv_1$ is a rainbow C_k . Thus, if $c(uv_1) \neq 1$ then $c(uv_1) \notin N_c(v_1)$.

Claim 2 $\{c(v_1 v_i) \mid i = 2, \dots, k-1\}$ is a set of distinct colors from $N_c(v_1)$ and $n_c(v_1) = k-2$.

From Claim 1 we see that the colors from $N_c(v_1)$ not equal to 1 appear only on edges $v_1 v_i$ for $i = 2, \dots, k-1$. Since $n_c(v_1) \geq k-2$, all these edges have distinct colors from $N_c(v_1)$ and $n_c(v_1) = k-2$.

Claim 3 For any $u \in V \setminus V_0$, $c(uv_{k-1}) \notin N_c(v_{k-1})$.

Assume otherwise, then $v_2v_3 \cdots v_{k-1}u$ is a good path. Then by Claim 2 $v_1v_3v_4 \cdots v_{k-1}uv_2v_1$ is a rainbow C_k .

Claim 4 $\{c(v_iv_{k-1}) \mid i = 1, \dots, k-2\}$ is a set of distinct colors from $N_c(v_{k-1})$ and $n_c(v_{k-1}) = k-2$.

By Claim 3, we see that all edges of colors from $N_c(v_{k-1})$ must occur on edges from $\{v_iv_{k-1} : i = 1, \dots, k-2\}$. Since $n_c(v_{k-1}) \geq k-2$, edges v_iv_{k-1} , $i = 1, \dots, k-2$ have distinct colors from $N_c(v_{k-1})$ and $n_c(v_{k-1}) = k-2$.

Claim 5 V_0 induces a rainbow complete subgraph with all colors unique in c . Moreover, for each v_i and each $u \notin V_0$, $c(uv_i)$ is not unique in c .

This claim follows from the above claims since for $i = 1, \dots, k-1$, $v_iv_{i+1} \cdots v_{k-1}v_1v_2 \cdots v_{i-1}$ is a good path, and $n_c(v_i) = k-2$.

Consider $u \notin V_0$ and a good path of length $k-2$ starting at u . Let the vertex set of this path be V_1 . If V_0 and V_1 share a vertex, say v_i , then v_iu has a unique color, a contradiction to Claim 5. Thus the graph is vertex-partitioned into copies of K_{k-1} each rainbow colored with unique colors. To avoid a rainbow C_k , any edges between two fixed parts must have the same color. Therefore $(k-1) \mid n$ and c is k -anticyclic. \square

By induction on n and the above lemma with $k = 4$, we have the following results.

Corollary 5. $AR(n, C_4) = |c^*| = 4/3n + O(1)$.

Proof. We need to show that for any edge-coloring c of K_n with no rainbow C_4 , $|c| \leq |c^*| = 4/3n + O(1)$.

We use induction on n . The statement trivially holds for $n = 3$. Let c be a coloring of $E[K_n]$ with no rainbow C_4 , $n \geq 4$. If for all $v \in V(K_n)$, $n_c(v) \geq 2$, then by Lemma 4, c is 4-anticyclic. So $|c| \leq |c^*|$. Suppose there is a $v \in V(K_n)$ with $n_c(v) \leq 1$. Let $G = K_n - v$. Let c' be the coloring of $E(G)$ induced by c . Then by induction hypothesis, $|c'| \leq 4/3(n-1) + O(1)$. Hence $|c| \leq |c'| + 1 \leq 4/3n + O(1)$. \square

Theorem 6. Let $n \geq 3$. Let G be a graph whose edges do not induce a star. Let $s = s(G)$ and $t = t(G)$ if G is bipartite.

$$\max R(n; G, C_4) = \begin{cases} \frac{4}{3}n + O(1), & \text{if } (\chi(G) = 2 \text{ and } s(G) \geq 4) \text{ or } (\chi(G) \geq 3) \\ n, & \text{otherwise} \end{cases}$$

Proof. Suppose $(\chi(G) = 2 \text{ and } s(G) \geq 4)$ or $(\chi(G) \geq 3)$. For the lower bound, consider the 4-anticyclic coloring c^* . Each color class of c^* is either $K_{1,m}$, $K_{2,m}$, or $K_{3,m}$ for some $m \geq 1$, thus c^* contains no monochromatic copy of G . The upper bound follows from Corollary 5.

Suppose G is bipartite and $s(G) \leq 3$. We use induction on n . The statement trivially holds for $n = 3$. Let c be a coloring of $E(K_n)$ with no monochromatic G and no rainbow C_4 . If $n_c(v) \geq 2$ for all $v \in V$, by Lemma 4 there is a color class of c that induces a $K_{3,3m}$ for some $m \geq 1$, which contains G . Hence we can find a $v \in V$ with $n_c(v) \leq 1$. Then by the induction hypothesis, $\max R(n; G, C_4) \leq n$. The lower bound is obtained from the coloring c^{**} with $s = s(G)$ and $k = 4$. Each color class of c^{**} is $K_{1,m}$ if $s(G) = 2$, either $K_{1,m}$ or $K_{2,m}$ if $s(G) = 3$ for some $m \geq 1$, thus c^{**} contains no monochromatic copy of G . The total number of colors in either cases is n . \square

APPENDIX A. PROOF OF CLAIM 2.4

Claim 2.4 Let $n, s \in \mathbb{N}$. Suppose n is sufficiently large and $s \geq 2$. Let $\xi_1, \dots, \xi_m \in \mathbb{N}$, $1 \leq \xi_i \leq s - 1$ and $\sum_{i=1}^m \xi_i \leq n$. Then

$$\sum_{i=1}^m \binom{\xi_i - 1}{2} \leq n \left(\frac{s-4}{2} + \frac{1}{s-1} \right).$$

The equality holds if and only if $m = \frac{n}{s-1}$ and $\xi_1 = \dots = \xi_m = s - 1$.

We use induction on m . If $m = 1$, then

$$\frac{(\xi - 1)(\xi - 2)}{2} \leq \frac{(s-2)(s-3)}{2} \leq n \left(\frac{s-4}{2} + \frac{1}{s-1} \right), \text{ for any } n \geq s - 1,$$

where the first inequality becomes equality iff $\xi = s - 1$, and the second does iff $n = s - 1$.

Suppose $m \geq 2$, $\sum_{i=1}^m \xi_i \leq n$, and $1 \leq \xi_i \leq s - 1$ for $1 \leq i \leq m$. Since $\sum_{i=1}^{m-1} \xi_i \leq n - \xi_m$, by induction,

$$\sum_{i=1}^{m-1} \binom{\xi_i - 1}{2} \leq (n - \xi_m) \left(\frac{s-4}{2} + \frac{1}{s-1} \right), \text{ for any } n \geq (m-1)(s-1) + \xi_m,$$

where the equality holds iff $m - 1 = \frac{n - \xi_m}{s-1}$ and $\xi_1 = \dots = \xi_{m-1} = s - 1$. Hence it is enough to show that $(n - \xi_m) \left(\frac{s-4}{2} + \frac{1}{s-1} \right) + \binom{\xi_m - 1}{2} \leq n \left(\frac{s-4}{2} + \frac{1}{s-1} \right)$ or $\xi_m \left(\frac{s-4}{2} + \frac{1}{s-1} \right) - \binom{\xi_m - 1}{2} \geq 0$, and the equality holds iff $\xi_m = s - 1$. If $\xi_m = 1$, that is obvious. Assume $\xi_m > 1$, then

$$\begin{aligned} \xi_m \left(\frac{s-4}{2} + \frac{1}{s-1} \right) - \binom{\xi_m - 1}{2} &= \xi_m \frac{(s-2)(s-3)}{2(s-1)} - \frac{(\xi_m - 1)(\xi_m - 2)}{2} \\ &= \frac{1}{2} \left(-\xi_m^2 + \left(s - 1 + \frac{2}{s-1} \right) \xi_m - 2 \right) = \frac{1}{2} \left(-\xi_m + \frac{2}{s-1} \right) (\xi_m - (s-1)) \geq 0, \end{aligned}$$

since $2 \leq \xi_m \leq s - 1$.

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