On the strong chromatic number of graphs

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Abstract

The strong chromatic number, $\chi_S(G)$, of an n-vertex graph G is the smallest number k such that after adding $k\lceil n/k\rceil - n$ isolated vertices to G and partitioning the vertices of the resulting graph into disjoint subsets $V_1, \ldots, V_{\lceil n/k\rceil}$ of size k each, one can find a proper k-vertex-coloring of the graph such that each part V_i, $i = 1, \ldots, \lceil n/k\rceil$, contains exactly one vertex of each color.

For any graph G with maximum degree Δ, it is easy to see that $\chi_S(G) \geq \Delta + 1$. Recently, Haxell proved that $\chi_S(G) \leq 3\Delta - 1$. In this paper, we improve this bound for graphs with large maximum degree. We show that $\chi_S(G) \leq 2\Delta$ if $\Delta \geq n/6$ and prove that this bound is sharp.

1 Introduction

An n-vertex graph G is **strongly r-colorable** if after adding $r\lceil n/r\rceil - n$ isolated vertices to G and partitioning the vertices of the resulting graph into disjoint subsets $V_1, \ldots, V_{\lceil n/r\rceil}$ of size r each, one can find a proper r-vertex-coloring of the graph such that each part V_i, $i = 1, \ldots, \lceil n/r\rceil$, contains exactly one vertex of each color. In [5], it was shown that if a graph G is strongly r-colorable, then it is strongly $(r + 1)$-colorable.

The **strong chromatic number** of G, denoted $\chi_S(G)$, is the smallest positive integer k such that G is strongly k-colorable.

The famous "cycle plus triangles" problem of Erdős [4], asking whether $\chi_S(C_{3m}) = 3$, was answered affirmatively by Fleischner and Stiebitz [7], [8], see also [13]. In general, Alon [1] proved that for any graph G with maximum degree Δ, $\chi_S(G) \leq c\Delta$, where c is a very large constant. Recently, Haxell [10] improved the bound by Alon drastically, proving that $\chi_S(G) \leq 3\Delta - 1$ for any graph G with maximum degree Δ.

As far as the lower bound is concerned, it is easy to see that the strong chromatic number of a graph with maximum degree Δ is at least $\Delta + 1$ by taking one of the V_i's to be the neighborhood of a vertex of maximum degree.

Let

$$f(\Delta, n) = \max\{\chi_S(G) : \quad G \text{ has maximum degree } \Delta \text{ and order } n\}.$$

Therefore, we have from the above results that

$$\Delta + 1 \leq f(\Delta, n) \leq 3\Delta - 1,$$

for any Δ and any $n \geq \Delta + 1$.

The following theorem is our main result which gives an exact value for $f(\Delta, n)$ when $\Delta \geq n/6$. It also provides a minimum degree condition for the existence of a K_3-factor in tripartite graphs.

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Theorem 1.1 *Let G be a graph on n vertices with maximum degree Δ, $\Delta \geq n/6$. Then $\chi_S(G) \leq 2\Delta$. Moreover, for any positive integers Δ and n, such that $\Delta \leq n/2$ there is a graph G_0 on n vertices, maximum degree Δ and $\chi_S(G_0) \geq 2\Delta$.*

Corollary 1.2 *For any positive integer Δ and any n such that $n/6 \leq \Delta \leq n/2$, $f(\Delta, n) = 2\Delta$. Moreover, $f(\Delta, n) \geq 2\Delta$ when $\Delta \leq n/2$.*

2 Proof of Theorem 1.1

Let $\Delta \leq n/2$, let G_0 be a graph formed by a disjoint union of a complete bipartite graph $K_{\Delta,\Delta}$ and $n - 2\Delta$ isolated vertices. As it was noted in [7], [8] and others, $\chi_S(G_0) \geq 2\Delta$. Indeed, assume that $\chi_S(G_0) \leq 2\Delta - 1$ then G is strongly r-colorable, for $r = 2\Delta - 1$. I.e., any partition of $V(G_0)$ and $r\lceil n/r\rceil - n$ isolated vertices into $t = \lceil n/r \rceil$ sets of equal sizes, $V_1, \ldots, V_t$, allows a proper r-coloring of G such that each V_i uses all the colors. Note that $t \geq \lceil 2\Delta/(2\Delta - 1)\rceil \geq 2$. Now, let A, B be the partite sets of a complete bipartite subgraph of G with $|A| = |B| = \Delta$, let $A \subseteq V_1$ and $B \subseteq V_2$. Then it is easy to see that it is impossible to find the desired r-coloring.

Together with the upper bound which we prove below, we shall have that $\chi_S(G_0) = 2\Delta$, when $n/6 \leq \Delta \leq n/2$.

Now we shall prove the main statement of Theorem 1.1 by providing an upper bound on the strong chromatic number. Let G be a graph on n vertices with maximum degree $\Delta \geq n/6$.

Let $\Delta \geq n/2$. Then $2\Delta \geq n$ and we trivially have that $\chi_S(G) \leq n \leq 2\Delta$.

Let $n/4 \leq \Delta < n/2$. In order to show that $\chi_S(G) \leq r$, for $r = 2\Delta$, we need to add $r\lceil n/r\rceil - n$ isolated vertices to G and partition the resulting vertex set arbitrarily into $t = \lceil n/r \rceil$ parts of sizes r, labelled $V_1, V_2, \ldots, V_t$. We have here that $\lceil 2\Delta/2\Delta \rceil < t \leq \lceil 4\Delta/2\Delta \rceil$, thus $t = 2$, and there are only two sets V_1, V_2. Each vertex in V_1 is nonadjacent to at least $|V_2|/2$ vertices in V_2 and vice versa. Consider the bipartite complement G' of this graph. That is, the edge set of G' consists of all pairs $\{v_1, v_2\}$, $v_1 \in V_1$ and $v_2 \in V_2$ such that $\{v_1, v_2\} \notin E(G)$. Applying the König-Hall theorem to G' gives a perfect matching, which provides a proper coloring of the original graph, G, with 2Δ colors, each represented exactly once in V_1 and exactly once in V_2.

Let $n/6 \leq \Delta < n/4$. As before, in order to verify that $\chi_S(G) \leq r$, for $r = 2\Delta$, we need to add $r\lceil n/r\rceil - n$ isolated vertices to G and partition the resulting vertex set arbitrarily into $t = \lceil n/r \rceil$ parts of sizes r, labelled $V_1, V_2, \ldots, V_t$. We have that $\lceil 4\Delta/2\Delta \rceil < t \leq \lceil 6\Delta/2\Delta \rceil$, thus $t = 3$. Let partition sets be V_1, V_2, V_3. In this case we prove the bound by extending partial colorings.

A **partial strong coloring** of G with respect to V_1, V_2, V_3 is a proper coloring of a subset of the vertices of G such that no two colored vertices in the same part V_i, $i = 1, 2, 3$ have the same color and each color class contains exactly 3 vertices. For a set S of vertices and a vertex coloring χ, we say that S is **partially multicolored** by χ if any two vertices in S, which are colored by χ, have distinct colors. Let χ be a maximal partial strong coloring of G with respect to V_1, V_2, V_3. We will show that we can always **enlarge** such partial strong coloring; i.e., create another partial strong coloring with more colors, until we color all the vertices. For a color c, we denote the vertices of this color $\{c_1, c_2, c_3\}$, where $c_i \in V_i$ for $i = 1, 2, 3$. We fix $v_1 \in V_1, v_2 \in V_2, v_3 \in V_3$ such that none of v_1, v_2, v_3 are colored by χ. For $i = 1, 2, 3$, define the following set:

$$X_i \stackrel{\text{def}}{=} \{u \in V_i : v_i \text{ is not adjacent to a vertex of color } \chi(u)\} \cup \{u \in V_i : u \text{ is not colored by } \chi\}.$$

Note that $|X_i| \geq |V_i| - \deg(v_i) \geq \Delta$, $i = 1, 2, 3$.

To simplify the notation, we shall assume that no color of χ is labelled by x, v, or w, we reserve x_i or w_i to denote a vertex in X_i (it might be colored or not colored), and v_i are the vertices fixed above. We shall write $z \sim y$, $z \not\sim y$ if $zy \in E(G)$, $zy \notin E(G)$, respectively. For disjoint subsets S_1, S_2 of vertices of G and a vertex z, $z \notin S_1$, we write $S_1 \sim S_2$ if each vertex in S_1 is adjacent to all vertices in S_2, $S_1 \not\sim S_2$ if there are no edges between S_1 and S_2, and $z \sim S_1$, $z \not\sim S_1$ if $\{z\} \sim S_1$, $\{z\} \not\sim S_1$, respectively.

To start the proof, we give two lemmas which allow us either to enlarge χ or to replace χ with another partial strong coloring such that some three specific vertices become uncolored and the number of colors remain the same.

Lemma 2.1 *Let $x_i \in X_i$, $i = 1, 2, 3$. Assume that one of the following holds:*

(1). $\{x_1, x_2, x_3\}$ is partially multicolored, or

(2). both (x_3 has the same color as exactly one of x_1 and x_2) and $v_3 \not\sim \{v_1, v_2\}$.

Then we can create a new coloring with as many color classes as χ and with x_i's being uncolored.

Proof. *(1):* Let $\{x_1, x_2, x_3\}$ be partially multicolored. We consider the cases of how many colors are assigned to x_1, x_2, x_3.

Suppose each x_i, $i = 1, 2, 3$ is colored; i.e., $x_1 = a_1$, $x_2 = b_2$, $x_3 = c_3$ with distinct colors a, b, c. Replace color classes a, b and c with new color classes $\{v_1, a_2, a_3\}$, $\{b_1, v_2, b_3\}$ and $\{c_1, c_2, v_3\}$, see Figure 1.

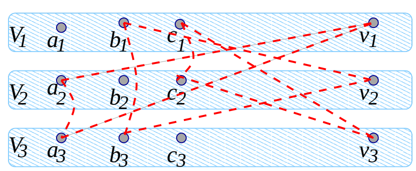

Figure 1: Color switches for Lemma 2.1

Now, suppose exactly one x_i is uncolored, without loss of generality, $x_1 = a_1$, $x_2 = b_2$ and x_3 is uncolored. Replace color classes a and b with new color classes $\{v_1, a_2, a_3\}$, $\{b_1, v_2, b_3\}$.

Finally, suppose exactly one x_i is colored, without loss of generality, $x_1 = a_1$ and x_2, x_3 are uncolored. Replace color class a with the new color class $\{v_1, a_2, a_3\}$.

Each case makes $\{x_1, x_2, x_3\}$ uncolored.

(2): Assume, without loss of generality, that $x_1 = a_1$, $x_3 = a_3$, for some color a. Replace color class a with $\{v_1, a_2, v_3\}$. If, in addition, x_2 is colored so that $x_2 = b_2$, then also replace color class b with $\{b_1, v_2, b_3\}$. This leaves x_1, x_2, x_3 uncolored. ■

Lemma 2.2 *Assume that one of the following holds:*

(1). there is a set $\{x_1, x_2, x_3\}$, with $x_i \in X_i$, $i = 1, 2, 3$, which induces an independent set and is partially multicolored, or

(2). there is a set $\{x_i, x_i', x_j, x_k\}$, with $x_i, x_i' \in X_i$, $x_j \in X_j$, $x_k \in X_k$, $\{i, j, k\} = \{1, 2, 3\}$ such that $\{x_j, x_k\}$ is partially multicolored, and both $\{x_i, x_j, x_k\}$ and $\{x_i', x_j, x_k\}$ induce independent sets.

Then the given partial strong coloring can be enlarged.

Proof. *(1)*: By Lemma 2.1 (1) there is a partial strong coloring with as many color classes as in χ and such that x_1, x_2, x_3 are uncolored. We can give these vertices a new color thus enlarging the coloring.

(2): If either $\{x_i, x_j, x_k\}$ or $\{x'_i, x_j, x_k\}$ is partially multicolored then we can use (1), otherwise assume, without loss of generality, that $i = 1, j = 2, k = 3$ and $x_1 = a_1$, $x'_1 = b_1$, $x_2 = b_2$, $x_3 = a_3$, for distinct colors a, b. We can find new color classes: $\{v_1, b_2, a_3\}$, $\{b_1, v_2, b_3\}$ and $\{a_1, a_2, v_3\}$ which replace color classes a, b and saturate vertices $\{v_1, v_2, v_3\}$, thus enlarging χ, see Figure 2.

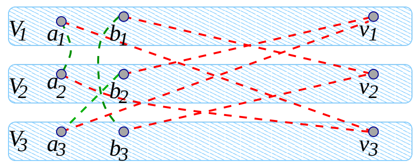

Figure 2: Color switches for Lemma 2.2

■

Next, we consider three cases depending on how many edges the set $\{v_1, v_2, v_3\}$ induces in G. We shall greedily choose appropriate $x_i \in X_i$, $i = 1, 2, 3$ and enlarge the coloring.

The proof begins with Case 1, where $\{v_1, v_2, v_3\}$ induces three edges. In this case, the coloring can be enlarged. In Case 2, $\{v_1, v_2, v_3\}$ induces two edges, without loss of generality $v_2 \not\sim v_3$, and either the coloring can be enlarged or another coloring with the same number of colors can be found so that there are three pairwise adjacent uncolored vertices reducing the analysis to Case 1. Finally, in Case 3 there is only one edge, without loss of generality v_1v_2, induced by $\{v_1, v_2, v_3\}$. In this case, either the coloring can be enlarged or we can find a coloring that puts us in Case 2 or Case 1. See Figure 3.

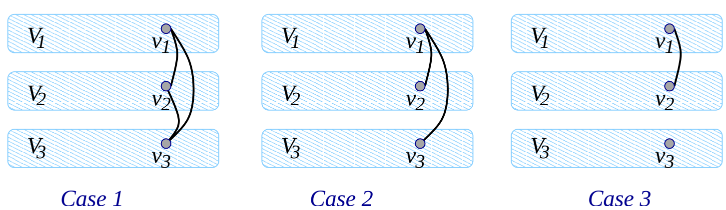

Figure 3: Cases 1, 2, 3

In Cases 1 and 2 we shall need the following parameter:

$$q \stackrel{\text{def}}{=} \max\{|N(x) \cap X_j| : \quad x \in X_i;\ i \neq j \text{ with } i, j \in \{1, 2, 3\}\}.$$

Case 1: $v_1 \sim v_2$, $v_1 \sim v_3$ and $v_2 \sim v_3$

We have $|X_i| \geq |V_i| - (\deg(v_i) - 2) \geq \Delta + 2$, for $i = 1, 2, 3$. Without loss of generality, assume that $q = |N(x_1) \cap X_2|$, for $x_1 \in X_1$. Let $x_2 \in X_2 \setminus N(x_1)$, be a vertex not of color $\chi(x_1)$. Consider

$S = X_3 \setminus (N(x_1) \cup N(x_2))$. By the choice of x_1, $|S| \geq |X_3| - (\Delta - q) - q \geq (\Delta + 2) - \Delta = 2$, thus there are two vertices $x_3, x_3' \in X_3$ nonadjacent to both x_1 and x_2. Therefore, Lemma 2.2 (2) can be applied to the four vertices x_1, x_2, x_3, x_3' to enlarge the coloring.

Case 2: $v_1 \sim v_2$, $v_1 \sim v_3$ and $v_2 \not\sim v_3$

In this case, $|X_1| \geq \Delta + 2$ and $|X_2|, |X_3| \geq \Delta + 1$. We say that (i, j) is a **best pair** if $q = |N(x_i) \cap X_j|$, for some $x_i \in X_i$. We shall consider sub-cases depending on which pair of indices is a best pair, see Figure 4.

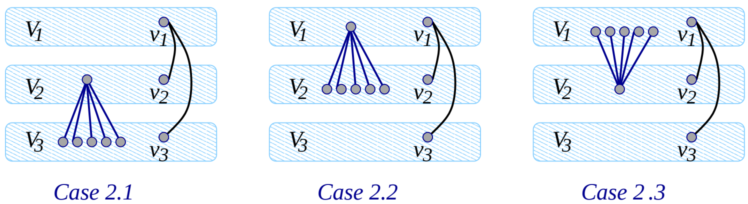

Figure 4: Subcases of Case 2

Case 2.1. (i, j) is a best pair with $i, j \in \{2, 3\}$.

Without loss of generality, let $i = 2$, $j = 3$, and $x_2 \in X_2$ such that $q = |N(x_2) \cap X_3|$. Consider $x_3 \in X_3$ such that x_3 is nonadjacent to x_2. Let x_1, x_1' be in X_1, such that $\{x_1, x_1'\} \not\sim \{x_2, x_3\}$. These vertices x_1, x_1' exist since $|X_1| \geq \Delta + 2$, $|N(x_2) \cap X_1| \leq \Delta - q$, and $|N(X_3) \cap X_1| \leq q$.

If $\{x_2, x_3\}$ is partially multicolored, we apply Lemma 2.2 (2) to x_1, x_1', x_2, x_3.

If $x_2 = a_2$, $x_3 = a_3$ for some color a and one of x_1, x_1', say x_1, is not colored, we replace a with two new color classes: $\{x_1, a_2, a_3\}$ and $\{a_1, v_2, v_3\}$.

Otherwise $x_2 = a_2$, $x_3 = a_3$ and, without loss of generality, $x_1 = b_1$, for $b \neq a$. In this case we replace a and b with three new color classes: $\{a_1, v_2, v_3\}$, $\{b_1, a_2, a_3\}$, $\{v_1, b_2, b_3\}$.

This enlarges the coloring.

Case 2.2. $(1, j)$ is a best pair with $j \in \{2, 3\}$ but neither $(2, 3)$ nor $(3, 2)$ is a best pair.

Assume, without loss of generality, that $j = 2$, and $x_1 \in X_1$ satisfies $q = |N(x_1) \cap X_2|$. We shall find additional vertices x_2, x_3, x_3' so as to apply Lemma 2.2 (2).

If $x_1 \not\sim v_2$, then choose $x_2 = v_2$. Otherwise, choose $x_2 \in X_2$ nonadjacent to x_1. If x_2 were colored the same as x_1, then our choice would give $x_1 = a_1$, $x_2 = a_2$ and $x_1 \sim v_2$. This means x_2 would not be in X_2. Thus $\{x_1, x_2\}$ is partially multicolored.

Next, choose $x_3, x_3' \in X_3$ such that $\{x_3, x_3'\} \not\sim \{x_1, x_2\}$. This is possible since $|X_3| \geq \Delta + 1$, $|N(x_1) \cap X_3| \leq \Delta - q$ and $|N(x_2) \cap X_3| \leq q - 1$ (if it were the case that $|N(x_2) \cap X_3| = q$, then we would have applied Case 2.1). Lemma 2.2 (2) can be applied to x_1, x_2, x_3, x_3' to enlarge the coloring.

Case 2.3. $(i, 1)$ is a best pair with $i \in \{2, 3\}$, but there is no other best pair.

Assume, without loss of generality, that $i = 2$ and $x_2 \in X_2$ such that $q = |N(x_2) \cap X_1|$. Let the vertex $x_1 \in X_1$ be nonadjacent to x_2 and either be uncolored or have a color different from that of x_2 if x_2 is colored. Such an x_1 exists since $|X_1| \geq \Delta + 2$. There are vertices $x_3, x'_3 \in X_3$ such that $\{x_3, x'_3\} \not\sim \{x_1, x_2\}$ since $|X_3| \geq \Delta + 1$, $|N(x_2) \cap X_3| \leq \Delta - q$, and $|N(x_1) \cap X_3| \leq q - 1$ (if it were the case that $|N(x_1) \cap X_3| = q$, then we would have applied Case 2.2). Lemma 2.2 (2) can be applied to x_1, x_2, x_3, x'_3 to enlarge the coloring.

Case 3: $v_1 \sim v_2$, $v_1 \not\sim v_3$ and $v_2 \not\sim v_3$

We show that in each of the Cases 3.1–3.3 one can enlarge the coloring, either directly or by finding a coloring with the same number of colors that satisfies either the conditions of Case 2 or the conditions of Case 1. These subcases are arranged according to the presence of specific paths in $X_1 \cup X_2 \cup X_3$, see Figure 5.

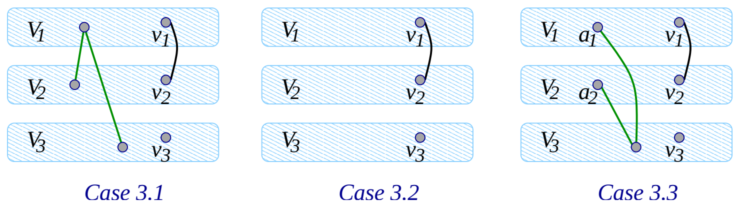

Figure 5: Subcases of Case 3

Case 3.1. There is a path P with three vertices w_1, w_2, w_3; $w_i \in X_i$, $i = 1, 2, 3$ such that either the vertices of P are partially multicolored or the middle vertex of P is in $X_1 \cup X_2$.

If P is partially multicolored, we can apply Lemma 2.1 (1) immediately to obtain a partial strong coloring with as many colors as χ and with vertices of P being uncolored. We can now choose $v_i = w_i$, $i = 1, 2, 3$ and use Case 2.

If P has repeated colors on its vertices, these can be only endvertices of the path. Without loss of generality, let the midpoint of P be $w_1 \in X_1$, let $a_2 = w_2$, $a_3 = w_3$ be the endpoints of P. We can then apply Lemma 2.1 (2) to create a new strong partial coloring with as many colors as in χ and with vertices of P being uncolored. This is Case 2.

Case 3.2. There is no path with three vertices w_1, w_2, w_3; $w_i \in X_i$, $i = 1, 2, 3$.

Let $G_{i,j}$ be the subgraph of G induced by the edges of G between X_i and X_j, with $i \neq j$ and $i, j \in \{1, 2, 3\}$. Note that $G_{i,j} = G_{j,i}$. Moreover, the distinct graphs $G_{i,j}$ are pairwise vertex-disjoint; otherwise, we have a forbidden path P. Let $Y_{i,j} = X_i \cap V(G_{i,j})$. Note that a vertex in $Y_{i,j}$ can be adjacent only to vertices in $Y_{j,i}$.

Claim a. If there is a nonedge xy in $G_{i,j}$ with endpoints in distinct sets X_i and X_j, $1 \leq i < j \leq 3$, then $i = 1, j = 2$ and $x = a_1$, $y = a_2$, for some color a.

Proof of Claim a. Suppose first that $j = 3$. Without loss of generality, let $i = 1$ and let $x_1 = x$, $x_3 = y$. We have that v_2 is not adjacent to either x_1 or x_3 since $X_2 \not\sim Y_{1,3} \cup Y_{3,1}$. Now, if $\{x_1, x_3\}$ is partially multicolored then $\{x_1, v_2, x_3\}$ is also partially multicolored. Apply Lemma 2.2 (1) with $\{x_1, v_2, x_3\}$ to enlarge χ. Otherwise, when $x_1 = a_1$ and $x_3 = a_3$ for some color a, we apply Lemma 2.2 (2) with $\{x_1, v_2, x_3, v_3\}$ (see that $a_1 = x_1 \not\sim v_3$ since $a_3 \in X_3$) to enlarge the coloring.

Let $j = 2$. Then $i = 1$, let $x_1 = x$ and $y_2 = y$. We have that v_3 is nonadjacent to both x_1 and y_2 because $X_3 \not\sim Y_{1,2} \cup Y_{2,1}$. If x_1 and y_2 fail to have the same color, then we apply Lemma 2.2 (1) with $\{x_1, y_2, v_3\}$ to enlarge the coloring. Otherwise, $x_1 = a_1$, $y_2 = a_2$ for some color a, as claimed. ■

Claim b. Both $X_1 \setminus Y_{1,2} \neq \emptyset$ and $X_2 \setminus Y_{2,1} \neq \emptyset$.

Proof of Claim b. Since $v_1 \in Y_{1,2}$, and v_1 is not colored by χ, we have by Claim a that v_1 is adjacent to every vertex in $Y_{2,1}$. But since $|X_2| \geq \Delta + 1$, we have that there is a vertex in X_2 nonadjacent to v_1. Thus, $Y_{2,1} \neq X_2$. A symmetric argument applied to v_2 gives that $Y_{1,2} \neq X_1$. ■

Consider v_3: If $v_3 \notin Y_{3,2}$ then fix some $x_2 \in X_2 \setminus Y_{2,1}$. Apply Lemma 2.2 (1) with the independent set $\{v_1, x_2, v_3\}$ to enlarge the coloring. If $v_3 \in Y_{3,2}$ then fix some $x_1 \in X_1 \setminus Y_{1,2}$. Apply Lemma 2.2 (1) with the independent set $\{x_1, v_2, v_3\}$ to enlarge the coloring.

Case 3.3. There is a path (w_1, w_3, w_2); with $w_i \in X_i$, for $i = 1, 2, 3$, and w_1, w_2 are of same color. Moreover, there are no paths satisfying the conditions of Case 3.1.

Let $w_1 = a_1$, $w_2 = a_2$. Observe that $a_1 \in X_1$ and $a_2 \in X_2$ implies that $v_1 \not\sim \{a_2, a_3\}$, $v_2 \not\sim \{a_1, a_3\}$.

Claim a. $v_3 \sim \{a_1, a_2\}$ and $v_3 \not\sim X_1 \cup X_2 \setminus \{a_1, a_2\}$.

Proof of Claim a. If $v_3 \not\sim a_1$, then $\{a_1, v_2, v_3\}$ is a partially multicolored independent set to which we can apply Lemma 2.2 (1) to enlarge the coloring. A symmetric argument establishes that $v_3 \sim a_2$.

Let $x_2 \in X_2 \setminus \{a_2\}$. If $v_3 \sim x_2$ then (a_1, v_3, x_2) is a path satisfying the conditions of Case 3.1. Thus $v_3 \not\sim x_2$. Similarly, $v_3 \not\sim x_1$, for any $x_1 \in X_1 \setminus \{a_1\}$. ■

Claim b. $N(v_2) = X_1 \setminus \{a_1\}$ and $N(v_1) = X_2 \setminus \{a_2\}$.

Proof of Claim b. Suppose that $v_1 \not\sim x_2$, for some $x_2 \in X_2 \setminus \{a_2\}$. Then $\{v_1, x_2, v_3\}$ is a partially multicolored independent set to which we can apply Lemma 2.2 (1) to enlarge the coloring.

A symmetric argument establishes that v_2 is adjacent to every vertex in $X_1 \setminus \{a_1\}$.

We have that $\deg(v_1), \deg(v_2) \leq \Delta$ and $|X_i| \geq \Delta + 1$, $i = 1, 2$. Since $N(v_1) \supseteq X_2 \setminus \{a_2\}$ and $N(v_2) \supseteq X_1 \setminus \{a_1\}$, we have that $N(v_1) = X_2 \setminus \{a_2\}$, $N(v_2) = X_1 \setminus \{a_1\}$, and, in particular, $|X_1| = |X_2| = \Delta + 1$. ■

Claim c. $X_3 \not\sim \{v_1, v_2\}$ and $X_3 \sim \{a_1, a_2\}$.

Proof of Claim c. Claim b immediately implies that $\{v_1, v_2\} \not\sim X_3$.

Let $x_3 \in X_3$ and suppose $x_3 \not\sim a_1$. If $\chi(x_3) \neq a$ then $\{a_1, v_2, x_3\}$ is a partially multicolored independent set, we can apply Lemma 2.2 (1) to enlarge the coloring. If $x_3 = a_3$

then we replace the color a with two new color classes: $\{a_1, v_2, v_3\}$ and $\{v_1, a_2, a_3\}$, thus enlarging the coloring.

A symmetric argument establishes that $x_3 \sim a_2$. ■

Note that $a_3 \notin X_3$.

Claim d. The vertices v_1, v_2, v_3 are the only uncolored vertices and every color class other than a has exactly one member in $X_1 \cup X_2$.

Proof of Claim d. Let b be a color used by χ, $b \neq a$, not present on vertices of X_1. By Claim b, $N(v_2) = X_1 \setminus \{a_1\}$, so $v_2 \not\sim b_1$ and $v_2 \not\sim b_3$. This implies that $b_2 \in X_2$. Thus, any color b, $b \neq a$, is used on some vertex in $X_1 \cup X_2$.

Let t be the number of uncolored vertices in each V_i, $i = 1, 2, 3$; i.e., the number of color classes in χ is $2\Delta - t$. The fact that each color class other than a contains at least one member of $X_1 \cup X_2$ and a contains two such members gives that $|X_1| + |X_2| \geq (2\Delta - t + 1) + 2t$. Here, the expression in parenthesis gives the lower bound on number of colored vertices in X_1 and X_2 and $2t$ is the number of uncolored vertices in X_1 and X_2. Because $|X_1| + |X_2| = 2\Delta + 2$, we have that $t = 1$. As a result, every vertex other than v_1, v_2, v_3 is colored and every color class other than a contains exactly one vertex from $X_1 \cup X_2$. ■

Having established these four claims, we will arrive at a contradiction to complete Case 3.3.

By Claim d, there are $2\Delta - 2$ colors different from a in χ. Let ν be the number of neighbors of v_3 colored differently than a. For a color c, $c \neq a$, the conditions $v_3 \not\sim c_1$ and $v_3 \not\sim c_2$ imply that $c_3 \in X_3$. Using also the fact that $v_3 \in X_3$, we have that

$$|X_3| \geq (2\Delta - 2 - \nu) + 1.$$

By Claim c, we have that $a_1 \sim X_3$, hence, $|X_3| \leq \Delta$. Using these inequalities, we have that $(2\Delta - 2 - \nu) + 1 \leq \Delta$, thus $\nu \geq \Delta - 1$. By Claim a, $v_3 \sim \{a_1, a_2\}$, thus we have that $\deg(v_3) \geq \nu + 2 \geq \Delta + 1$, a contradiction. This concludes Case 3.3, and the proof of Theorem 1.1. ■

3 Concluding Remarks

It should be noted that Theorem 1.1 is equivalent to the following:

Corollary 3.1 *Let G be a tripartite graph with parts of size n each. If the minimum degree of G is at least $3n/2$ then G has a K_3-factor.*

This result provides another sufficient condition for the existence of K_3-factors. For other results in this area, see for example, [3, 9, 2, 6, 11, 12].

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