A note on graph coloring extensions and list-colorings

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Abstract

Let G be a graph with maximum degree $\Delta \geq 3$ not equal to $K_{\Delta+1}$ and let P be a subset of vertices with pairwise distance, $d(P)$, between them at least 8. Let each vertex x be assigned a list of colors of size Δ if $x \in V \setminus P$ and 1 if $x \in P$. We prove that it is possible to color $V(G)$ such that adjacent vertices receive different colors and each vertex has a color from its list. We show that $d(P)$ cannot be improved. This generalization of Brooks' theorem answers the following question of Albertson positively: If G and P are objects described above, can any coloring of P in at most Δ colors be extended to a proper coloring of G in at most Δ colors?

We say that a vertex-coloring of a graph $G = (V, E)$ is *proper* if the colors used on adjacent vertices are distinct. For an assignment of a color set (typically called a list) $l(x)$ to each vertex $x \in V$, we say that vertices are *colored from their lists* by a coloring c if $c(x) \in l(x)$ for each $x \in V$; c is called a *list-coloring* of G. A coloring c of $V(G)$ *extends* a coloring c' of vertices in P if it is a proper coloring with $c(x) = c'(x)$ for each $x \in P$. We denote by $d_G(x)$ the degree of x in a graph G and by $G[X]$ the subgraph of G induced by a set of vertices X.

The classic Brooks' theorem states that any simple connected graph G with maximum degree Δ can be colored properly in at most Δ colors unless $G = K_{\Delta+1}$ or G is an odd cycle. Recently, Albertson posed the following question. Take a graph described above, precolor a fixed set of vertices P in Δ colors arbitrarily. Under what condition on P can we extend that coloring to a proper coloring of G in at most Δ colors? He asks whether this condition is a large distance between the vertices in P. Albertson noticed though, that the maximum degree of a graph should be at least three. Indeed, it is easy to see that one cannot obtain a proper coloring of a path with an even number of vertices in two colors if the end-points are precolored in the same color. Here, we show that if the maximum degree is at least three, then there is a positive answer to Albertson's question when the pairwise distance, $d(P)$, between vertices of P is at least 8; moreover, this distance is optimal. The color extension problem is closely related to the concept of a list-coloring of graphs. Indeed, we can reformulate Albertson's question the following way. For set $S = \{1, \cdots, \Delta\}$, let the vertices of P be assigned lists of single colors from S and let every other vertex be assigned list S. Can G be properly list-colored from these lists if $d(P)$ is large enough? We answer this question by presenting a more general result. Our main tool is a corollary of the theorem about list-coloring of hypergraphs by Kostochka, Stiebitz and Wirth [4] which was also investigated independently by Borodin. The list-coloring version of Brooks' theorem was considered much earlier by Vizing [5]. We need a couple of definitions first. A *block* containing an edge e is a maximum 2-connected subgraph containing that edge or an edge e itself if such 2-connected subgraph does not exist. A *separating vertex* in a block is a vertex whose deletion disconnects the graph, i.e., a cutvertex of a graph. An *end-block* is a block with exactly one separating vertex. A *Gallai tree* is a graph all of whose blocks are either complete graphs, odd cycles, or single edges.

Theorem 1 (Kostochka, Stiebitz, Wirth). *Let $G = (V, E)$ be a connected graph. For each $x \in V$, let $l(x)$ be an assigned list of colors, $|l(x)| \geq d(x)$. If G is not list-colorable from these lists then it is a Gallai tree and $|l(x)| = d(x)$ for each $x \in V$.*

Figure 1 depicts graphs illustrating the exactness of our results. Next we give a formal description of graph G_1 from the figure.

A general construction Consider Δ copies of $K_{\Delta+1} \setminus e$, say $B_1, \cdots, B_\Delta$, where the deleted edge of B_i is $u_i v_i$ for each $i = 1, \cdots, \Delta$. Let B be a complete graph on vertices $w_1, \cdots, w_\Delta$. Then G_1 is formed from a disjoint union of $B, B_1, \cdots, B_\Delta$ and edges $u_1 w_1$, $u_2 w_2, \cdots, u_\Delta w_\Delta$. It is easy to see that the maximum degree of G_1 is Δ and G_1 is not equal to $K_{\Delta+1}$. Assign a list $\{1\}$ to each vertex in P and a list $\{1, \cdots, \Delta\}$ to every other vertex. Then, under any Δ-coloring c of B_is from the corresponding lists, $c(u_i) = c(v_i) = 1$. Thus $c(w_i) \neq 1$ for all $i = 1, \cdots, \Delta$. Since we need Δ colors for B, all different from 1, we need at least $\Delta + 1$ colors altogether to color G_1.

Theorem 2. *Let G be a graph with maximum degree $\Delta \geq 3$, not equal to $K_{\Delta+1}$. Let $P \subseteq V$, $d(P) \geq 8$. Let vertices in P and $V \setminus P$ be assigned arbitrary lists of sizes 1 and Δ respectively. Then G can be properly colored from these lists.*

Proof of Theorem 2. For each $x \in V$, let $l(x)$ be an assigned list of colors. The general idea of the proof is to list color all copies of $K_{\Delta+1} \setminus e$ in G which share a vertex of degree $\Delta - 1$ with P and then use Theorem 1 to list-color the rest. Let G have copies $B_1, \cdots, B_t$ of $K_{\Delta+1} \setminus e$ with $u_i v_i$ be the deleted edge, $u_i \in P$ for each $i = 1, \cdots, t$. Note that all B_is are vertex disjoint.

First we treat the case when $\Delta \geq 4$. When $\Delta = 3$ we need some more details to be considered separately. We shall color vertices of all B_is from their lists. For each $i = 1, \cdots, t$ we delete $l(u_i)$ from the lists of vertices in $B_i - \{u_i, v_i\}$ obtaining lists of size at least $\Delta - 1$. The degree of each vertex in $B_i - u_i$ is $\Delta - 1$; moreover, the new lists have size at least $\Delta - 1$ on $V(B_i) - \{u_i, v_i\}$ and Δ on v_i. Thus, by Theorem 1 we can properly

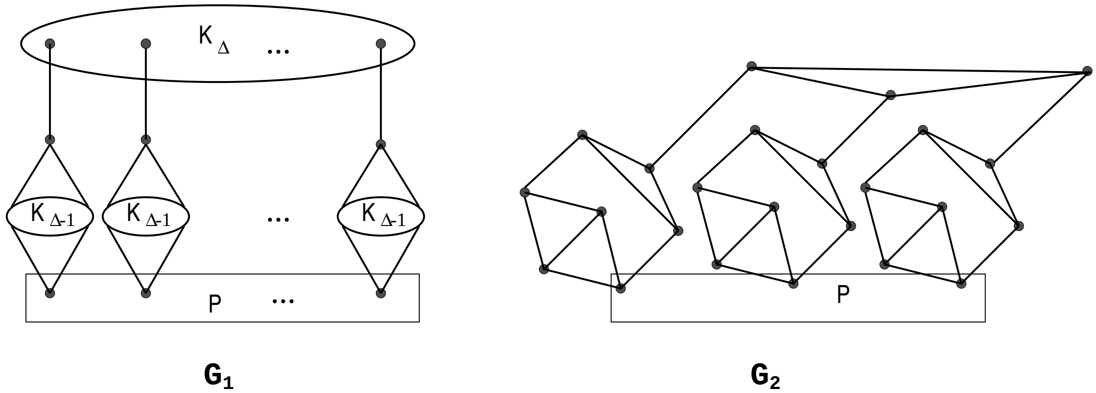

Figure 1: Two graphs with maximum degree Δ, which are not properly colorable from the list $\{1, \cdots, \Delta\}$ assigned to all vertices of $V \setminus P$ and the list $\{1\}$ assigned to all vertices of P.

color $B_i - u_i$ from the above lists, obtaining a proper coloring of B_i from the original lists. Let a_i be a color of v_i under some such coloring for each $i = 1, \cdots, t$.

Now, we consider a new graph G_1 obtained from G by deleting $V(B_i) - \{u_i, v_i\}$. Let $P_1 = P \cup \{v_1, \cdots, v_t\}$. Note that G_1 does not have copies of $K_{\Delta+1} \setminus e$ sharing a vertex of degree $\Delta - 1$ with P_1, and each vertex u_i or v_i for $i = 1, \cdots, t$ is adjacent to at most one vertex in G_1. Now, we need to color G_2 induced by $V(G_1) \setminus P_1$. We assign the new lists to $V(G_2)$ as follows.

$$l_2(x) = \begin{cases} l(x) \setminus l(u_i) & \text{if } xu_i \in E(G),\ xv_i \notin E(G), \\ l(x) \setminus \{a_i\} & \text{if } xv_i \in E(G),\ xu_i \notin E(G), \\ l(x) \setminus (\{a_i\} \cup l(u_i)) & \text{if } xu_i, xv_i \in E(G), \\ l(x) \setminus l(p) & \text{if } xp \in E(G),\ p \in P \setminus \{u_1, \cdots, u_t\}. \end{cases}$$

Note that if $x \in V(G_2)$ is adjacent to more than one vertex of P_1, these vertices must be u_i and v_i for some i, so only one of the above cases can hold. Assume that G_2 is not properly colorable from the lists l_2. Then, by Theorem 1 it is a Gallai tree with $d_{G_2}(x) = |l_2(x)|$ for each $x \in V(G_2)$. Thus, $d_{G_2}(x) = \Delta$, $\Delta - 1$ or $\Delta - 2$ when x is not adjacent to any vertex in P_1, when it is adjacent to one or two such vertices respectively. Thus each vertex in G_2 has degree at least 2.

We may assume that G_2 is connected since we can color the connected components separately. Let B be an end-block with a separating vertex x (if such exists) of G_2. B is a complete graph, or an odd cycle; moreover, $|V(B)| \geq 3$. If $B = G_2$ there must be an edge between $V(B)$ and P_1 since G is connected, if $B \neq G_2$ there is an edge between $V(B)$ and P_1 since $d_B(x) < d_{G_2}(x)$. Let uv be an edge of B. If $up, vq \in E(G)$ with $p, q \in P_1$, then either $p = q$ or $\{p, q\} = \{u_i, v_i\}$ for some i, otherwise the distance condition will be violated. Moreover, since $d_{G_1}(u_i) \leq 1$ and $d_{G_1}(v_i) \leq 1$ for each $i = 1, \cdots, t$, we have that all vertices of $B - x$ (or B if $G_2 = B$) are adjacent to the same vertex $p \in P$, and

$p \notin \{u_1, \cdots, u_t\} \cup \{v_1, \cdots, v_t\}$. Therefore $d_{G_2}(v) = \Delta - 1$ for each $v \in V(B - x)$, (or for each $v \in V(B)$ if $G_2 = B$), i.e., $B = K_\Delta$. But then $V(B) \cup \{p\}$ induces $K_{\Delta+1} \setminus e$ if $B \neq G_2$, a contradiction to the way we constructed G_1 or, if $B = G_2$, $V(B) \cup \{p\}$ induces $K_{\Delta+1}$ a contradiction to the condition of the theorem.

Now we treat the case when $\Delta = 3$. Assume, without loss of generality, that there are indices $1 \leq s' < s \leq t$, vertices w_i, $i = 1, \cdots, s$ and triangles $T_i = w_i w_i' w_i''$, $i = s'+1, \cdots, s$ such that w_i is adjacent to both u_i and v_i for $i = 1, \cdots, s'$, and $w_i' u_i, w_i'' v_i \in E(G)$ for $i = s' + 1, \cdots, s$. Note that all these w_i's are distinct. For each $i = 1, \cdots, s'$ let L_i be induced by $V(B_i)$ and w_i, for each $i = s'+1, \cdots, s$, let L_i be induced by $V(B_i)$ and $V(T_i)$, and, finally, for each $i = s + 1, \cdots, t$ let $L_i = B_i$. We properly color each L_i, $i = 1, \cdots, t$ from the original lists $l(x)$ and assume that w_i gets the color b_i for $i = 1, \cdots, s$ and v_i gets the color a_i for $i = s + 1, \cdots, t$.

We create G_1 from G by deleting vertices of $L_i - w_i$ for all $i = 1, \cdots, s$ and vertices of $B_i - \{u_i, v_i\}$ for $i = s + 1, \cdots, t$. Let $P_1 = (P \cap V(G_1)) \cup \{w_1, \cdots, w_s\} \cup \{v_{s+1}, \cdots, v_t\}$. Now, consider G_2, the subgraph of G_1 induced by $V(G_1) \setminus P_1$. Note that each vertex in G_2 has at most one neighbor in P_1, otherwise we violate the distance condition. Again, we create new lists for $l_2(x)$ for each vertex x of G_2 as follows.

$$l_2(x) = \begin{cases} l(x) \setminus l(u_i) & \text{if } xu_i \in E(G), \\ l(x) \setminus \{a_i\} & \text{if } xv_i \in E(G), \\ l(x) \setminus \{b_i\} & \text{if } xw_i \in E(G), \\ l(x) \setminus l(p) & \text{if } xp \in E(G),\ p \in P,\ p \neq u_i, v_i, \text{or } w_i \text{ for any } i \in \{1, \cdots, t\}. \end{cases}$$

Assume now that G_2 is not colored properly from the lists l_2. Then, by Theorem 1, we have $d_{G_2}(x) = |l_2(x)| = 3$ or 2. If G_2 is a block B, then it must be an odd cycle with all vertices adjacent to some vertices in P_1. It is easy to see that then all the vertices of G_2 must be adjacent to the same $p \in P_1$. In this case, we have $B \cup p$ induce K_4, a contradiction. If G_2 has a cut-vertex, let B be an end-block with a separating vertex x. B must be an odd cycle, either with all vertices in $B - x$ being adjacent to the same vertex in P and resulting in $K_4 \setminus e$, or with $V(B) - x = \{y, z\}$, where y and z are adjacent to u_i and v_i respectively for some i. In this case we get $B = K_3$ and $V(B_i) \cup V(B)$ induce a graph isomorphic to some L_j, a contradiction to the way we constructed G_2.

□

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