Collection of quiz Problems for MATH 307

CHAPTER 1

Problem 1. Let $\mathbf{x} = (2,3)$, $\mathbf{y} = (4,6)$, $\mathbf{z} = (-3,2)$.
a)(3 points) Draw $-\mathbf{x} + \frac{1}{2}\mathbf{y} + \mathbf{z}$.
b)(3 points) Are the following vectors parallel: $\mathbf{x} - \mathbf{y}$ and $2\mathbf{z} + \mathbf{y}$? Explain.

Problem 2 (5 points) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $c, d \in \mathbb{R}$. Prove that $(\mathbf{x} - c\mathbf{y})d = d\mathbf{x} - dc\mathbf{y}$.

Problem 3. (5 points) Solve the following system of linear algebraic equations.

$$\begin{cases} -z + x + y = 0 \\ 8x + 2y - 6z = 2 \\ -4x - y + 3z = -1. \end{cases} \tag{1}$$

Problem 4. (4 points each) Let l be a line in $\mathbb{R}^3$ with the following parametric equation:

$$\begin{pmatrix} y \\ z \\ x \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R} \tag{2}$$

Sketch this line in the provided coordinate system.

Problem 5 (3 points)
Give an example of a system of linear algebraic equations in variables x_1 and x_2 which has no solutions.

Problem 6. Given the following system of linear algebraic equations.
a) (3 points)Write an augmented matrix corresponding to the system.
b) (6 points) Use method of Gaussian elimination to find reduced row-echelon form.
c) (2 points) Circle all leading ones.
d) (3 points) Write the solution of the system.

$$\begin{aligned} x_1 + 5x_3 + 6x_4 &= 3 \\ x_1 + 3x_2 + 5x_3 + 9x_4 &= 3 \\ x_2 + 5x_3 - x_4 &= 0 \end{aligned} \tag{3}$$

Problem 7. (5 points) Solve the following system of linear algebraic equations.

$$\begin{cases} -z + x + y = 0 \\ 8x + 2y - 6z = 2 \\ -4x - y + 3z = -1. \end{cases} \quad (3)$$

Problems 1 (5 points)
Let $\mathbf{a_1} = (1, 1, 0, -1)$, $\mathbf{a_2} = (1, 0, 0, -1)$, $\mathbf{a_3} = (0, 0, 1, -1)$, $\mathbf{b} = (1, 2, 3, 4)$. Is $\mathbf{b}$ in the $Span(\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\})$? If yes, give the corresponding coefficients. If not, justify your answer.

Problem 2 (1 point each)True/False?
a) Any linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is has the form $a\mathbf{v_1} - b\mathbf{v_2} + c\mathbf{v_3}$, where $a, b, c \in \mathbb{R}$.

b) Any linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is in the $Span(\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\})$.

c) A product of 4×10 and 4×3 matrices is well-defined.

d) $\mathbb{R}^3$ is a span for some set of vectors.

Problem 8. (4 points each) Let l be a line in $\mathbb{R}^3$ with the following parametric equation:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} t + \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \quad (3)$$

Sketch this line (label the graph properly).

Problem 9 Give an example of a system of linear algebraic equations in variables x_1 and x_2 which has no solutions.

Problem 10. (5 points) Find the reduced row echelon form for the augmented matrix of the following system. Use this to write all solutions.

$$\begin{cases} x_1 + x_2 = 0 \\ x_1 - 3x_2 = 0 \\ x_1 - x_2 = 2. \end{cases} \quad (3)$$

Problem 11. (7 points) Find the reduced row echelon form for the augmented matrix of the following system. Use this to write all solutions.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 - x_3 + 3x_4 = 0 \\ x_1 - 2x_2 + x_3 + x_4 = 9. \end{cases} \tag{3}$$

Problem 1. Given the following system of linear algebraic equations.
a) (3 points)Write an augmented matrix corresponding to the system.
c) (2 points) Circle all leading elements.

$$\begin{aligned} x_1 + 5x_3 + 6x_4 &= 3 \\ x_1 + 3x_2 + 5x_3 + 9x_4 + 2 &= 3 \\ x_2 + 5x_3 - x_1 &= 0 \end{aligned} \tag{1}$$

Problem 12. For each of the matrices below:
a) (1 point per matrix) cicle all leading elements
b) (1 point per marix) write to the left of the matrix:
"RRE" if the matrix is in the reduced row-echelon form,
"RE" if the matrix is in the row-echelon, but not reduced row echelon form,
"N" if the matrix is not in row-echelon form.

$$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 3 & 0 & 7 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1. \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1. \end{pmatrix},$$

$$\begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

Problem 13 Perform one elementary row operation on each of the matrices below such that the element in position $(2,3)$ becomes zero. Write what operation you are doing.

a) (2 points)

$$\begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

b) (2 points)

$$\begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 2 & 0 \end{pmatrix}$$

Problem 14. Given the following system of linear algebraic equations.
a) (3 points)Write an augmented matrix corresponding to the system.
b) (6 points) Use method of Gaussian elimination to find reduced row-echelon form.
c) (2 points) Circle all leading ones.
d) (3 points) Write the solution of the system.

$$\begin{aligned} x_1 + 5x_3 + 6x_4 &= 3 \\ x_1 + 3x_2 + 5x_3 + 9x_4 &= 3 \\ x_2 + 5x_3 - x_4 &= 0 \end{aligned} \tag{-6}$$

Problem 15. For each of the following augmented matrices corresponding to the systems of linear algebraic equations with unknowns $x_1, \cdots, x_n$ in order, find the **number of solutions**, **is the matrix in reduced row-echelon form (T/F)?**

a) (3points) $\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$

b) (3points) $\left(\begin{array}{cccc|c} 1 & 2 & 4 & 5 & 6 \\ 0 & 1 & 4 & 5 & 6 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

Problem 16 (3 points) Can a system of linear algebraic equations have exactly three solutions?

Problem 17. (3 points each) Let the following matrices be augmented matrices for the systems of linear algebraic equations with unknowns $x_1, x_2, \ldots$. Give the solution sets for each of the system:

$$A_1 = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Problem 18. (6 points) Find the reduced row echelon form for the augmented matrix of the following system.

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ 2x_1 + x_2 - x_3 + 3x_4 = 0 \\ x_1 - 2x_2 + x_3 + x_4 = 9. \end{cases} \tag{-9}$$

Problem 19. (3 points) Sketch the following vector $\mathbf{q} = 2\mathbf{u} - 2\mathbf{v} + \frac{1}{3}\mathbf{w}$.

Problem 20. For each of the following augmented matrices corresponding to the systems of linear algebraic equations with unknowns $x_1, \cdots, x_n$ in order, find the **number of solutions**, **is the matrix in reduced row-echelon form (T/F)?**

a) (3points)
$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 1 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

b) (3points)
$$\left(\begin{array}{cccc|c} 1 & 2 & 4 & 5 & 6 \\ 0 & 1 & 4 & 5 & 6 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Problem 21 (3 points) Can a system of linear algebraic equations have exactly three solutions?

Give a geometric explanation in case when there are two variables involved.

Problem 24. (6 points) Consider the following vectors.
$\mathbf{v_1} = (1, 2, 3, 1, 1), \mathbf{v_2} = (1, 2, 0, 0, 0), \mathbf{v_3} = (0, 0, 0, 1, 1),\ \mathbf{v_4} = (0, 1, 1, 0, 0),$ $\mathbf{v_5} = (0, 0, 1, 0, 0)$.

a) Determine whether they are linearly independent or not.
b) If they are linearly dependent, give the coefficients of the corresponding linear combination.

Problem 25

For each set of the vectors below, determine whether they are linearly dependent or not without performing any calculations. (hint: you can use the matrix or other criteria for linear independence). Provide your explanation.

a) (2 points)
$\mathbf{v_1} = (1,2,3), \mathbf{v_2} = (1,2,0), \mathbf{v_3} = (1,1,0), \mathbf{v_4} = (0,1,0)$.

b) (2 points)
$\mathbf{v_1} = (1,2,3,4,5), \mathbf{v_2} = (0,0,0,0,0), \mathbf{v_3} = (0,0,0,1,1), \mathbf{v_4} = (0,1,1,0,0)$.

c) (2 points)
$\mathbf{v_1} = (1,2,3,4,5), \mathbf{v_2} = (2,4,6,8,10)$.

Problem 26. Consider the following vectors.
$\mathbf{v_1} = (1,2,3,1,1), \mathbf{v_2} = (1,2,0,0,0), \mathbf{v_3} = (0,0,0,1,1),\ \mathbf{v_4} = (0,1,1,0,0)$, $\mathbf{v_5} = (0,0,1,0,0)$.
a) (4 points) Is $\mathbf{v_1}$ in the $Span(\{\mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\})$?
b) (4 points) If yes, provide the coefficients of corresponding linear combination.
c) (4 points) Is $Span(\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} = Span(\{\mathbf{v_3}, \mathbf{v_4}, \mathbf{v_1}\})$? Justify your answer.

Problem 28 (3 points) Find all the solutions (and write in a set theoretic notation) for the following system with unknowns $x_1, x_2, x_3, x_4, x_5, x_6$.

$$\begin{cases} x_2 + x_3 - 5x_5 = 0 \\ x_1 + 3x_6 = 5 \\ x_4 - 2x_6 = 4. \end{cases} \tag{-9}$$

Problem 30 (3 points) How many solutions does the system with the following augmented matrix have? What is the rank of this matrix?

1	2	4	5
0	0	1	3
0	0	0	0

Problem 31 (3 points) Let T be a linear transformation with matrix

1	2	0
0	0	1
0	1	0

Find $T(\mathbf{x})$ where $\mathbf{x} = (1,2,0)$. Sketch $\mathbf{x}$ and $T(\mathbf{x})$.

Problem 32 (6 points) Let

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \mathbf{x}.$$

Find all vectors $\mathbf{x}$ such that they are not changed by transform T, i.e., that image of $\mathbf{x}$ under T is $\mathbf{x}$.

Problem 33 (6 points) Are the following functions linear transforms? Justify your answer.
a) $T((x_1, x_2, x_3)) = (x_2 - x_1,\ x_3 + 4x_1)$.
b) $T((x_1, x_2, x_3)) = (x_1 - x_2 + 3,\ x_3 - x_1)$.

Problem 34.(5points) Sketch $T(P)$, where P is the region below and T is the following linear transform.

$$T(\mathbf{x}) = \begin{matrix} 1 & 2 \\ -1 & 1 \end{matrix} \mathbf{x}$$

Problem 35 (6 points) Let T be a linear transform. $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $T(\mathbf{e_1}) = (1, 2, 0, 1)$, $T(\mathbf{e_2}) = (2, 2, 0, 0)$, $T(\mathbf{e_3}) = (1, 0, 0, 7)$, $T(\mathbf{e_4}) = (1, 0, -1, 0)$.

Find $T((4, 3, 0, 6))$ and the matrix of the transform.

Problem 36.(10 points) Let T_1 be a linear transformation corresponding to first rotation clockwise by 30 degrees. Let T_2 be projection with respect to the vector $\mathbf{w} = (3, 4)$. Find the matrices of these transforms.

Problem 37 (6 points) Let

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 0 & -1 \\ 4 & 0 & -2 \end{pmatrix} \mathbf{x}.$$

Find all vectors $\mathbf{x}$ such that they are not changed by transform T.

Problem 38 (6 points) Let T be a linear transform such that
$T((1, 2, 3)) = (0, 2, 1), \quad T((1, 0, 3)) = (1, 2, 4), \quad T((0, 0, 1)) = (1, 1, -1)$.
a) Find a matrix A of T.
b) Sketch the image of the following region under T.

Problem 39 (4 points) Let T be a transform such that $T((0, 0, 0, 0)) = (1, 2)$.
a) Find domain and codomain of T.
b) Prove that T is not linear.

Problem 40 (6 points) Verify that the following transform is linear by formally checking the linearity properties.

$T((x_1, x_2, x_3)) = (x_2 - x_1,\ 2x_1,\ x_3 + 4x_1,\ x_3)$.

Problem 41. Consider the following vectors.
$\mathbf{v_1} = (1, 0, 3, 1, 1), \mathbf{v_2} = (1, 2, 0, 0, 0), \mathbf{v_3} = (0, 0, 0, 1, 1),\ \mathbf{v_4} = (0, 1, 1, 0, 0)$, $\mathbf{v_5} = (0, 0, 1, 0, 0)$.
Are these vectors linearly dependent or linearly independent? If they are linearly dependent, provide the corresponding coefficients.

Problem 42 (6 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transform corresponding to rotation by $\pi/6$ clockwise.

Is this transform one-to-one (give a complete argument)? Justify your answer.

CHAPTERs 2, 3

Problem 22 (4 points) Calculate the following matrix product.

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \qquad \text{(-9)}$$

Problem 23 (2points) Is product of 2×4 and 4×100 matrices defined? (Y/N)

Problem 27 Calculate the product of the following matrices:

a) (2 points)

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \text{(-9)}$$

b) (3 points)

$$\begin{pmatrix} 1 & 2 & 0 \\ 4 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad \text{(-9)}$$

Problem 29. Find determinant of each of the following matrices: (simplify first)

a) (3 points)

1	3	400	1	6
2	6	412	5	7
-1	-3	179	5	0

b) (3 points)

1	2	3	4
5	6	7	8
1	1	0	1
0	0	0	1

Problem 57 (4 points) Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} \qquad \text{(-9)}$$

Use Cramer's rule to find solution of the following system: $A\mathbf{x} = (1, 0, -1)^T$.

Problem 58 (4 points) Let S be a triangle with vertices $(0,0), (1,0), (0,2)$. Let T be a linear transform with matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \tag{-9}$$

Find the area of $T(S)$ using determinants.

Problem 59. (8 points) Find the determinant of the following matrix:

$$\begin{pmatrix} a & 2a & 0 & 3a & 4a & 3a \\ 1 & 2 & 1 & 5 & 6 & 5 \\ b & 3b & 0 & b & 2b & b \\ -b & 0 & -2b & b & b & b \\ 0 & a & a & a & a & a \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Problem 60 (2 points) Let A be an n by n matrix, and $\mathbf{b}$ be a vector with n components. Let $det(A) = 3$, $det(A_i(\mathbf{b})) = i$, $i = 1, 2, \ldots, n$. Solve the system $A\mathbf{x} = \mathbf{b}$.

Problem 61 (4 points) Express the determinant of the following matrix as a sum of determinants of 4×4 matrices using splitting by 4th column. (DO NOT EVALUATE those determinants) (DO NOT simplify)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 7 & 8 & 9 & 10 & 11 \\ 13 & 14 & 15 & 16 & 17 \\ 19 & 20 & 21 & 22 & 23 \\ 25 & 26 & 27 & 28 & 29 \end{pmatrix}.$$

Problem 43 (5 points) Find the following matrix product.

$$\begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 4 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Problem 44. (6 points) Find inverse of the following matrix or show that it does not exist

$$A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 1 & -2 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{pmatrix}.$$

Problem 50 (6 points) Find the determinant of the following matrix:

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & -2 \\ 1 & 3 & 0 & -1 \end{pmatrix} \qquad \text{(-12)}$$

Problem 56 (2 points each part) Let A be a 4×4 matrix such that $det(A) = 7$.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \quad B = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 2a_{21} - a_{11} & 2a_{22} - a_{12} & 2a_{23} - a_{13} & 2a_{24} - a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$$C = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \qquad D = \begin{pmatrix} a_{11} & a_{13} & a_{12} & a_{14} \\ a_{21} & a_{23} & a_{22} & a_{24} \\ a_{31} & a_{33} & a_{32} & a_{34} \\ a_{41} & a_{43} & a_{42} & a_{44} \end{pmatrix}$$

Find the following:
a) $det((A^3)^{-1})$
b) $det(-3A)$
c) $det(B)$
d) $det(C)$
e) $det(D)$.

Problem 57 (4 points) Let

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 0 \end{pmatrix} \qquad \text{(-14)}$$

Use Cramer's rule to find solution of the following system: $A\mathbf{x} = (1, 0, -1)^T$.

Problem 58 (4 points) Let S be a triangle with vertices $(0,0),(1,0),(0,2)$. Let T be a linear transform with matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \qquad \text{(-14)}$$

Find the area of $T(S)$ using determinants.

Problem 59. (8 points) Find the determinant of the following matrix:

$$\begin{pmatrix} \text{a} & \text{2a} & 0 & \text{3a} & \text{4a} & \text{3a} \\ 1 & 2 & 1 & 5 & 6 & 5 \\ \text{b} & \text{3b} & 0 & \text{b} & \text{2b} & \text{b} \\ \text{-b} & 0 & \text{-2b} & \text{b} & \text{b} & \text{b} \\ 0 & \text{a} & \text{a} & \text{a} & \text{a} & \text{a} \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

Problem 60 (2 points) Let A be an n by n matrix, and $\mathbf{b}$ be a vector with n components. Let $det(A) = 3$, $det(A_i(\mathbf{b})) = i$, $i = 1, 2, \ldots, n$. Solve the system $A\mathbf{x} = \mathbf{b}$.

Problem 61 (4 points) Express the determinant of the following matrix as a sum of determinants of 4×4 matrices using splitting by 4th column. (DO NOT EVALUATE those determinants) (DO NOT simplify)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 7 & 8 & 9 & 10 & 11 \\ 13 & 14 & 15 & 16 & 17 \\ 19 & 20 & 21 & 22 & 23 \\ 25 & 26 & 27 & 28 & 29 \end{pmatrix}.$$

CHAPTER 4

Problem 45 (5 points) Let $S = \{(a+b, a-b, c+3b) : a, b, c \in \mathbb{R}\}$. Is S a subspace? Show ALL your work.

Problem 46 (4 extra-points) Let $S = \{(a+b+1, a-b-2, c) : a, b, c \in \mathbb{R}\}$. Is S a subspace? Show ALL your work.

Problem 47. (8 points) Let

$$T(\mathbf{x}) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -2 \\ 1 & 3 & 4 & 0 \\ 3 & 5 & 5 & 5 \end{pmatrix} \mathbf{x}.$$

Find $Ker(T)$ and $Im(T)$.

Problem 48 (4 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transform. It is known that $Im(T) \neq \mathbb{R}^2$ and that $(1,2) \in Im(T)$. Find and sketch $Im(T)$.

Problem 49 (6 points) Find the inverse of the following matrix or show that it does not exits.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 3 & 0 & -1 \end{pmatrix} \tag{-16}$$

Problem 51. (8 points) Find the dimension of the following subspace:

$$S = Span(\{(1, 1, -1, 3, 4), (0, -1, 1, -3, -3), (2, 0, 0, 0, 2)\}).$$

Problem 52 Let A be a 10×6 matrix of rank 3.
a)(2 points) What is the dimension of $Null(A)$?
b)(2 points) Let $L(\mathbf{v}) = A\mathbf{v}$, be a linear transform. Find $dim(Im(L))$.

Problem 53 Let S be a subspace having an ordered basis $B = ((1, 1, 0, 1, 1), (-1, 0, 1, 1, 0))$. Let $\mathbf{x} = (2, 7, 5, 12, 7)$.
a) (3 points) Find the coordinatization $[\mathbf{x}]_B$ of $\mathbf{x}$ with respect to B.
b) (2 points) Let $[\mathbf{y}]_B = (-1, 3)$. Find $\mathbf{y}$.

Problem 54. (6 points) Let $S = \{(1, 2, -1, 0), (1, 3, 4, 0), (3, 8, 7, 0), (1, 1, 1, 1)\}$. Is S a basis of $\mathbb{R}^4$? Show all your calculations.

Problem 55 (6 points) Let $S = \{(1, 3, 2), (2, 0, 1)\}$. Is S a basis of the following subspace $V = \{\mathbf{x} \in \mathbb{R}^3 : x_1 + x_2 - 2x_3 = 0\}$? Show all your calculations.

Problem 62. (6 points)
Find a basis of $V = Span(\{(1, 2, -1, 0), (0, 0, 0, 0), (1, 3, 4, 0), (3, 2, 7, 0), (1, 1, 1, 0)\})$.
(use whatever method you'd like)

Problem 63 (6 points)
Find a basis B of $\mathbb{R}^3$ such that B contains a vector $(1, 3, 2)$.
(If you are not using any specific method, justify your answer)
(Note that row-reduction method will not necessarily give you the desired result).

Problem 64

a)(4 points)

Let $H = \{(x + 3y, -9y - 3x) : x, y \in \mathbb{R}\}$.

Is H a subspace of some vector space V? If yes, prove it and state what is V. If no, prove it.

b)(4 points)

Let $H = \{y^2 : y \in \mathbb{R}\}$.

Is H a subspace of $\mathbb{R}$?

c) (3 points)

Let H be a set of polynomials of the following form:

$H = \{bx^{2a} : b \in \mathbb{R}, a \in \mathbb{N}\}$.

For example H contains $x^2, 4x^4, \sqrt{5}x^{10}, \ldots$. Is H a vector space over $\mathbb{R}$ with usual operations of function addition and multiplication by a scalar? Justify your answer.

Problem 65. (6 points)

Let $B = (1, 2, 3, 4), (0, 0, 1, -1), (0, 0, 1, 1), (1, 0, 0, 2))$.

a) Prove that B is an ordered basis of $\mathbb{R}^4$.

b) Find transition matrix T_B such that $\mathbf{w} = T_B[\mathbf{w}]_b$ for any $\mathbf{w} \in \mathbb{R}^4$.

c) Find $[\mathbf{w}]_B$ for $\mathbf{w} = (1, 0, 2, -4)$.

Problem 66. (6 points)

Let $B = (1, 2, 3, 4), (0, 0, 0, -1), (0, 0, 1, 1), (1, 0, 0, 2))$.

a) Check that B is an ordered basis of $\mathbb{R}^4$.

b) Find transition matrix T_B such that $\mathbf{w} = T_B[\mathbf{w}]_b$ for any $\mathbf{w} \in \mathbb{R}^4$.

c) Find $[\mathbf{w}]_B$ for $\mathbf{w} = (1, 1, 2, -4)$.

Problem 68 (6 points)

Let $B = ((1, 2, 3), (0, 1, 0), (-1, 0, 0))$ and $C = ((1, 0, 0), (4, 0, 1), (1, 1, 0))$.

Let $L : \mathbb{R}^{\nvDash} \to \mathbb{R}^3$ given by

$L((x_1, x_2, x_3)) = (x_3 - x_1, \quad x_2 + 5x_1, \quad -x_3 + x_1 + x_2)$.

Find a matrix A_{BC} such that

$$[L(\mathbf{x})]_C = A_{BC}[\mathbf{x}]_B.$$

Problem 69 (6 points)

Let $H = \{(a - b - c + d, -a + b + c, d, a - b - c + 2d) : a, b, c, d \in \mathbb{R}\}$.

Find a basis of H.

Problem 73 (3 points each subproblem)

Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 4 & 7 & 8 \\ 2 & 4 & 2 & 8 & 15 & 7 \\ 45 & 90 & 2 & 180 & 316 & 1 \end{pmatrix}.$$

It is given that

$$ref(A) = \begin{pmatrix} 1 & 2 & 0 & 4 & 7 & 8 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Find (and write down your argumentation)
a) A basis for $Null(A)$.
b) A basis for $Col(A)$.
c) Dimensions of $RowSpace(A)$, $Col(A)$, $Null(A)$.
d) Does $A\mathbf{x} = \mathbf{b}$ have a solution for any $\mathbf{b}$?

Problem 75.
Let $S = Span(\{t^2, 2t^3 - 1, 30, \cos(t), t^3 - t^2, \cos(t) + t\})$.
a) (6 points) Find an ordered basis B for S. Justify your answer.
b) (3 points) Find a coordinatization of the following function with respect to B: $f(t) = 7\cos(t) + 5t^3 - 4$.
c) (2 points) Find $dim(S)$.

CHAPTER 5

Problem 76. (12 points)
For the following matrix, find:
1) the eigenvalues
2) corresponding eigenspaces
3) a specific eigenvector for each eigenvalue.

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 \end{pmatrix}$$

CHAPTER 6

Problem 70 (6 points)
Let P be a plane spanned by vectors $(1,2,3)$ and $(3,4,0)$. Does the following set B of vectors form a basis of P?
Justify your work.
$B = \{(-2,-2,3),(1,0,-6)\}$.

Problem 71. (6 points)
a) Prove that the following is an orthonormal basis of a vector space V.
b) What is the dimension of that vector space?
$B = \{(1/\sqrt{3}, 1\sqrt{3}, -1/\sqrt{3}), (0, 1\sqrt{2}, 1/\sqrt{2})\}$.

Problem 72 Let $\mathbf{v} = (1,2,3,4,5,6,7,8) \in \mathbb{R}^8$.
Let $V = Span(\{(3,0,0,0,0,0,0,0),(0,1,1,0,0,0,0,0),(0,0,0,0,-4,3,0,0),(0,0,0,0,0,0,0,2)\}$.
a) Find the projection of $\mathbf{v}$ onto V.
b) Find the orthogonal complement of V.

Problem 74. (12 points)
Let $V = \{(x_1,x_2,x_3,x_4) : 2x_1 + x_2 - x_3 + x_4 = 0\}$. Find a basis B of V and use Gram-Schmidt orthogonalization to find an orthonormal basis of V.

GENERAL

Problems 1 (5 points)
Let $\mathbf{a_1} = (1, 1, 0, -1)$, $\mathbf{a_2} = (1, 0, 0, -1)$, $\mathbf{a_3} = (0, 0, 1, -1)$, $\mathbf{b} = (1, 2, 3, 4)$. Is $\mathbf{b}$ in the $Span(\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}\})$? If yes, give the corresponding coefficients. If not, justify your answer.

Problem 2 (1 point each) True/False?
a) Any linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is has the form $a\mathbf{v_1} - b\mathbf{v_2} + c\mathbf{v_3}$, where $a, b, c \in \mathbb{R}$.

b) Any linear combination of vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ is in the $Span(\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\})$.

c) A product of 4×10 and 4×3 matrices is well-defined.

d) $\mathbb{R}^3$ is a span for some set of vectors.