Bases for subspaces and how to find them.

Definition. Let S be a subspace of $\mathbb{R}^n$. A set B of vectors is a basis for S if
1. $Span(B) = S$,
2. B is linearly independent.

Fact 1. A basis of S is a largest collection of linearly independent vectors in S.

Fact 2. A basis of S is a smallest collection of vectors spanning S.

Fact 3. All bases of S have the same number of vectors.

Row-reduction method to find a basis.

Let $S = Span(\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\})$.
-Arrange $\mathbf{u_i}$s as rows of a matrix, call it A.
-Find $rref(A)$.
-The set of nonzero rows of $rref(A)$ forms a basis of S.

Column selection method to find a basis.

Let $S = Span(\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\})$.
-Arrange $\mathbf{u_i}$s as columns of a matrix, call it A.
-Find $rref(A)$.
-Identify pivot columns of $rref(A)$.
-Select corresponding columns of A as a basis for S.

Enlarging a given set of vectors to a basis of S.

Let $S = Span(\{\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}\})$. Let $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m} \in S$ be a given set of linearly independent vectors.

Find a basis of S containing $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$.
-Arrange $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_m}$, $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ as columns of a matrix in THIS order.
-Apply column selection method to find a basis.
-Since $\mathbf{u_i}$s are linearly independent, they will correspond to pivot columns and thus will be selected in a basis.

Examples

Row-reduction Let $S = Span\{(1,2,3,4),(0,-1,2,1),(1,0,0,5)\}$. Find a basis for S.

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 1 & 2 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 2 & 3 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 7 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 7 & 1 \end{bmatrix}$$

Answer: A basis of S is $B = \{(1,0,0,5),(0,-1,2,1),(0,0,7,1)\}$.

Column-selection Let $S = Span\{(1,2,3,4),(1,1,5,5),(0,-1,2,1),(1,0,0,5)\}$. Find a basis for S.

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & -1 & 0 \\ 3 & 5 & 2 & 0 \\ 4 & 5 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 2 & 2 & -3 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now, the column selection method chooses columns number $1, 2, 4$ from the original matrix.

Answer: A basis of S is $B = \{(1,2,3,4),(1,1,5,5),(1,0,0,5)\}$.

Enlarging a given set of vectors to a basis.

Let $S = Span\{(1,2,3,4),(0,-1,2,1),(1,0,0,5)\}$. Let $\mathbf{v_1} = (1,1,5,5), \mathbf{v_2} = (0,2,-4,-2)$. Find a basis of S containing $\mathbf{v_1}$ and $\mathbf{v_2}$.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 2 & 2 & -1 & 0 \\ 5 & -4 & 3 & 2 & 0 \\ 5 & -2 & 4 & 1 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 & -1 \\ 0 & -4 & -2 & 2 & -5 \\ 0 & -2 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & -7 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, the column selection method chooses columns number $1, 2, 5$.

Answer: A basis of S is $B = \{(1,1,5,5),(0,2,-4,-2),(1,0,0,5)\}$.