Mathematics 503 Problem Set 5

Due Friday, May 2, 2003

1. Do **Exercise 8.4**, page 456.

2. Consider the nonlinear boundary-value problem

$$\begin{aligned} u'' &= 15(u^2 - x^2)u', \qquad -1 < x < 0, \\ u(-1) &= 0.96 \\ u(0) &= 0.001. \end{aligned}$$

This problem has three isolated solutions, and the corresponding initial value problem with $u(-1) = 0.96$, $u'(-1) = s$ is not very sensitive to the value of s. Derive the variational equation to be used with Newton's method to determine the values of s that will satisfy the second boundary condition. Use the single shooting method to find all three solutions. Graph the solutions that you find.

3. Consider the problem

$$\begin{aligned} u''' &= 2u'' + u' - 2u, \qquad 0 < x < b, \\ u(0) &= 1, \\ u(b) - u'(b) &= 0, \\ u(b) &= 1. \end{aligned}$$

Let $b = 100$.

(a) Convert the differential equation to a first order system

$$y' = Ay, \qquad B_0 y(0) + B_b y(b) = \beta.$$

Find the fundamental solution satisfying $Y(0) = I$.

(b) Use this exact fundamental solution to construct (approximately) the system $Qs = g$. Show that the resulting Q is extremely ill-conditioned and that consequently s cannot be accurately obtained by the single shooting method.

(c) Show that the boundary problem itself is well-conditioned. Let $\Psi(x)$ be the well-scaled fundamental solution

$$\Psi(x) = \begin{pmatrix} e^{-x} & e^{x-b} & e^{2(x-b)} \\ -e^{-x} & e^{x-b} & 2e^{2(x-b)} \\ e^{-x} & e^{x-b} & 4e^{2(x-b)} \end{pmatrix}.$$

Construct the matrix $Q_0 = B_a\Psi(a) + B_b\Psi(b)$ and show that Q_0 is well-conditioned.

(d) Solve the problem numerically for $b = 1, 10$, and 100. Use the multiple shooting method. What are your conclusions?