Mathematics 503 Problem Set 3

Due Friday, March 28, 2003

1. A four-stage Runge-Kutta formula is displayed in the Butcher tableau as follows.

$$\begin{array}{c|cccc}
0 & & & & \\
c_2 & c_2 & & & \\
c_3 & c_3 - a_{32} & a_{32} & & \\
1 & c_3 - a_{42} - a_{43} & a_{42} & a_{43} & \\
\hline
 & b_1 & b_2 & b_3 & b_4
\end{array}$$

Retrieve the Maple text file `rk4.txt` and MATLAB M-file `rk4.m` from the `Examples` directory. You can use them to express the coefficients of a four-stage Runge-Kutta formula of order four in terms of c_2 and c_3

Derive a particular 4th order Runge-Kutta formula by choosing c_2 and c_3 that approximately minimize some measure of the 5th order truncation error such as

$$\|T_5\|_2 \doteq \left(\sum_{|\tau|=5} \alpha_\tau^2 \left(\frac{1}{\tau!} - b^T A^{(\tau)} \right)^2 \right)^{1/2}$$

or

$$\|T_5\|_\infty \doteq \max_{|\tau|=5} \alpha_\tau \left| \frac{1}{\tau!} - b^T A^{(\tau)} \right|$$

Observe that the truncation error coefficients can be regarded as functions of (c_2, c_3) since the formula coefficients (A, b, c) are. The nine rooted trees of order 5 are displayed in Table 4.3 on page 150 of the textbook.

It's not necessary or even desirable to perform an "exact" optimization. Instead, look for example at a contour plot of $\|T_5\|$ in the (c_2, c_3) plane and choose c_2 and c_3 to be "simple" numbers that lead to "simple" values for the coefficients.

2. Implement the Runge-Kutta formula you derived in Problem 1 as a MATLAB function M-file

```
function [x,y] = RK4FixH(dfun,xspan,y0,nsteps)
```

and demonstrate its performance by integrating the Van der Pol equation over the interval $0 \le x \le 20$:

$$\begin{aligned} y_1' &= y_2 & \qquad y_1(0) &= 2 \\ y_2' &= -y_1 + (1 - y_1^2) \cdot y_2 & \qquad y_2(0) &= 0. \end{aligned}$$

3. Implement stepsize control with an error estimate by extrapolation as described in class and in **Example 5.11** in the text, in a MATLAB function M-file

```
function [x,y] = RK4ex(dfun,xspan,y0,Atol,Rtol)
```

and demonstrate its performance by solving the Arenstorff orbit problem described in **Exercise 4.4**, page 184 of the text. You can modify the ODE file `orbitode.m` that comes with MATLAB: it already has the differential equations; you need only reset a few things such as the parameter μ, the initial conditions and the integration period.