Mathematics 503 Problem Set 1

Due Monday, February 3, 2003

By submitting a paper for this assignment you declare that you are not submitting any unattributed work of any other person.

1. Let $(x_0, \ldots, x_n)$ be distinct points. Recall the unit polynomials for the Lagrange form of the interpolating polynomial

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}.$$

Prove that $\displaystyle\sum_{k=0}^{n} L_k(x) = 1$ for all x.

Let $\omega_n(x) = (x - x_0) \cdots (x - x_n)$ and observe that

$$\omega_n'(x_k) = (x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n).$$

Then show that the Lagrange formula can be written

$$p_n(x) = \sum_{k=0}^{n} \frac{\omega_n(x)}{(x - x_k)\omega_n'(x_k)} \cdot f(x_k).$$

Next, since the divided difference $f[x_0, \ldots, x_n]$ is the leading coefficient in $p_n(x)$, use the Lagrange formula written this way to show that

$$f[x_0, \ldots, x_n] = \sum_{k=0}^{n} \frac{f(x_k)}{\omega_n'(x_k)}.$$

Finally, deduce from this last formula that the divided difference is invariant under permutations of the x_k.

2. Let $p_n(x)$ be the polynomial that interpolates a function f on the *equally spaced* points $(x_0, x_{-1}, \ldots, x_{-n})$. Let h be the step between consecutive points and write $s = (x - x_0)/h$.

The *backward difference* operator ∇ is defined by $(\nabla f)_k = f_k - f_{k-1}$. Express p_n in terms of backward differences in two ways:

(a) Express the divided differences $f[x_0, \ldots, x_{-k}]$ in terms of the backward differences $\nabla^k f_0$. Then substitute into the Newton divided difference formula for $p_n(x)$.

(b) Use the Calculus of Finite Differences: using the forward shift operator E you have $f(x) = f(x_0 + sh) = E^s f_0$. The backward difference operator is $\nabla = I - E^{-1}$. Deduce that $E^s = (I - \nabla)^{-s}$, and expand using the binomial series.

3. Calculate four approximations to e^x on the interval $-1 \leq x \leq 1$ using cubic polynomials:

(a) Taylor approximation about $x = 0$.

(b) Minimax approximation.

(c) Least squares approximation by Legendre polynomials.

(d) Least squares approximation by Chebyshev polynomials.

Graph the errors in the four approximations and compare.