Mathematics 502 Problem Set 2

1. Here is the output from the script to solve the batting average problem. There are two integer solutions. Figure 1 shows them along with the solution of the rounded equations. The relative change in the solution is within the bounds of the relative change in the matrix and right hand side multiplied by the condition number.

```
>> BattingAve
Rounding the equations doesn't work:
x =  24,  y= 134,  Average Before: 0.179  After 0.207

Search finds 2 solution(s).

cond(A) =  74.17

  H  AB  AVE<  AVE> ||du||/||u|| k(A)*||dA||/||A|| k(A)*||db||/||b||
 24 132 0.182 0.210   0.011798        0.011967          0.017246
 25 137 0.182 0.210   0.025760        0.027044          0.024964
```

And here is script `BattingAve.m`.

```
% BattingAve script solves the batting average problem.
close all

avbef = 0.182; avaft = 0.210;

% Rounded equations
A0 = [1 -avbef; 1 -avaft];
b0 = [0; 6*avaft-5];
x0 = A0 \ b0; xr = round(x0);
av0 = xr(1)/xr(2); av1 = (xr(1)+5)/(xr(2)+6);

disp('Rounding the equations doesn''t work:')
disp(sprintf('x = %3d,  y= %3d,  Average Before: %5.3f  After %5.3f', ...
             xr(1),xr(2),av0,av1))

% Plot the two pairs of lines surrounding the solution(s)
y = [129 139]; delta = 5e-4;

x1lo = (avbef-delta)*y;
x1hi = (avbef+delta)*y;

x2lo = (avaft-delta)*(y+6)-5;
x2hi = (avaft+delta)*(y+6)-5;

plot(x1lo,y,'r',x1hi,y,'r',x2lo,y,'b',x2hi,y,'b')
xlabel('Hits'), ylabel('Times at Bat')
```

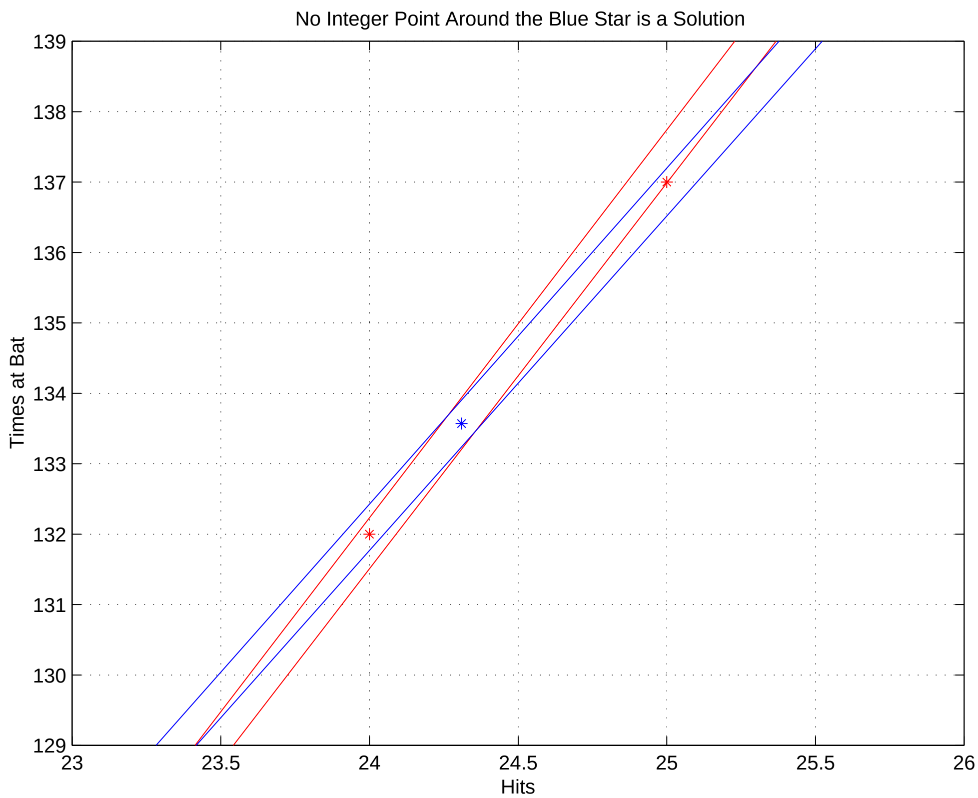

Figure 1: From BattingAve.m

```
title('No Integer Point Around the Blue Star is a Solution')
grid

hold on
plot(x0(1),x0(2),'b*')

% Check 22 integer points for solutions
s = [ 24*ones(1,11) 25*ones(1,11);
      129:139       129:139       ];
I = find( abs(     s(1,:)./s(2,:)     - avbef ) < delta & ...
          abs( (s(1,:)+5)./(s(2,:)+6) - avaft ) < delta );

%     If solutions found, record, plot and display.

if ~isempty(I)
  ns = length(I);
  s = s(:,I);
  plot(s(1,:),s(2,:),'r*')
  disp(sprintf('\nSearch finds %d solution(s).\n',ns))
end
% Save solutions found, print a table entry for each one.
kA = cond(A0);
% Print a header
disp(sprintf('cond(A) = %6.2f\n',kA))
normA = norm(A0);
normb = norm(b0);
normu = norm(x0);
disp('  H  AB  AVE<  AVE> ||du||/||u|| k(A)*||dA||/||A|| k(A)*||db||/||b||')
for n = 1:ns
  x = s(1,n); y = s(2,n);
  ndu = norm(s(:,n)-x0);
  av0 = x/y; av1 = (x+5)/(y+6);
  A = [1 -av0; 1 -av1]; ndA = norm(A-A0);
  b = [0; av1*6-5];     ndb = norm(b-b0);
  disp(sprintf( ...
   '%3d %3d %5.3f %5.3f   %8.6f        %8.6f          %8.6f', ...
   x, y, x/y, (x+5)/(y+6), ndu/normu, kA*ndA/normA, kA*ndb/normb))
end
```

2. Two MATLAB functions from the book *Introduction to Scientific Computing* by Charles Van Loan show one way to solve this problem. They can be retrieved from the `Examples` directory on the course web site. Function `CholTrid` performs the Cholesky factorization of a symmetric tridiagonal matrix represented using two vectors. Function `CholTridSol` uses the factorization to solve the linear system.

The session diary demonstrating the solution follows.

```
>> d = 2*ones(64,1);         % diagonal of matrix
>> e = -[0;ones(63,1)];      % sub- and super-diagonal
>> [g,h]=CholTrid(d,e);      % Bidiagonal Cholesky factor
>> b=ones(64,1);             % Right side vector
>> x=CholTridSol(g,h,b);     % Solution
>> n=1:64; plot(n,x),grid    % Graph solution components

%  Calculate the norm of the residual for this x.

>> norm(-[0;x(1:63)] + 2*x - [x(2:64);0] - b)
ans =
   5.7190e-13
```

3. If A is $m \times n$ with $m < n$ and A has rank m, then the null space N of A is an $n - m$ dimensional subspace of $\mathbf{R}^n$. If b is in the column space of A, and x_p is the minimum-norm solution of $Ax = b$, then the set of *all* solutions of $Ax = b$ is $x_p + N$, and $x_p \perp N$. This is the theoretical basis for the solution algorithms sketched below.

Normal Equations Since x_p is orthogonal to the null space of A, it follows that x_p belongs to the column space of A^T, say $x_p = A^T y$. Thus $AA^T y = b$ is the normal equation. Matrix AA^T is symmetric and positive definite; solve for y and compute x_p.

QR Factorization Let $Q \cdot R$ be the qr-factorization of A^T. The first m columns of Q span the column space of A^T, that contains x_p. Partition $Q = [Q_1\ Q_2]$ with Q_1 containing the first m columns. We are done if we find y in $\mathbf{R}^m$ so that $x_p = Q_1 y$ satisfies $Ax_p = b$. Now, $A = R^T Q^T$, so this last equation reads

$$[\hat{R}^T\ 0] \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} Q_1 y = b,$$

or, $\hat{R}^T y = b$ ($\hat{R}$ is the square upper triangular matrix formed of the first m rows of R). Solve for y by forward substitution.

SVD Let $U \Sigma V^T$ be the singular value decomposition of A^T. Then U is $n \times m$ and its columns span the column space of A^T, so we set $x_p = Uy$ similar to before. The equation $Ax = b$ is then $V \Sigma U^T U y = b$. Since $U^T U$ is the $m \times m$ identity matrix, and V is $m \times m$ orthogonal this equation reduces to $\Sigma y = V^T b$.