AN EXAMPLE OF GAUSSIAN QUADRATURE

In this example we work out the one- and two-point Gaussian quadrature formulas for improper integrals of the form

\[ \int_0^1 f(x) \frac{dx}{\sqrt{x}}. \]

We will apply the derived formulas to approximate the integral

(1) \[ \int_0^1 e^{-x} \frac{dx}{\sqrt{x}} = \sqrt{\pi} \text{erf}(1) \approx 1.493648. \]

1. INNER PRODUCT, ORTHOGONAL POLYNOMIALS

The nodes of Gaussian quadrature formulas are the roots of certain orthogonal polynomials, so as a preparatory step we work out the first few orthogonal polynomials.

Incorporate the \( x^{-1/2} \) singularity into an inner product for functions on \([0,1]\) by defining

(2) \[ \langle f, g \rangle = \int_0^1 f(x)g(x) \frac{dx}{\sqrt{x}}. \]

Next, we derive orthogonal polynomials \( Q_0(x) \), \( Q_1(x) \), \( Q_2(x) \) by applying the Gram-Schmidt algorithm of §6.8. First set

(3) \[ Q_0(x) = 1. \]

Next, get \( Q_1 \) by subtracting from \( x \) its component along \( Q_0 \):

(4) \[ Q_1(x) = x - \frac{\langle xQ_0 \rangle}{\langle Q_0Q_0 \rangle}Q_0 = x - \frac{2/3}{2} \cdot 1 = x - 1/3. \]

Finally, \( Q_2 \) is obtained similarly from \( x^2 \):

\[ Q_2(x) = x^2 - \frac{\langle x^2Q_0 \rangle}{\langle Q_0Q_0 \rangle}Q_0(x) - \frac{\langle x^2Q_1 \rangle}{\langle Q_1Q_1 \rangle}Q_1(x) \]
\[ = x^2 - \frac{2/5}{2} \cdot 1 - \frac{16/105}{8/45} \cdot (x - 1/3) \]
\[ = x^2 - \frac{6}{7}x + \frac{3}{35}. \]
2. Quadrature Nodes and Weights

2.1. One-point formula. The one-point quadrature formula uses for node $x_{10}$ the sole root of $Q_1(x) = x - \frac{1}{3}$, so $x_{10} = \frac{1}{3}$. The weight $b_{10}$ can be found by the method of undetermined coefficients. The weight must be chosen so that the formula integrates the function $1$ exactly:

$$ b_{10} \cdot 1 = \int_0^1 \frac{dx}{\sqrt{x}} = 2. $$

This Gaussian formula integrates the function $x$ exactly too:

$$ b_{10} \cdot x_{10} = 2 \cdot \frac{1}{3} = \int_0^1 \frac{x \, dx}{\sqrt{x}}. $$

2.2. Two-point formula. The two-point quadrature formula uses for nodes the roots of $Q_2(x)$ from (5):

$$ x_{20} = \frac{3}{7} - \frac{2}{35} \sqrt{30}, \quad x_{20} = \frac{3}{7} + \frac{2}{35} \sqrt{30}. $$

The weights can again be found by the method of undetermined coefficients, since the formula must integrate the functions $1$ and $x$ exactly. Alternatively, we compute them from the Lagrange unit polynomials.

$$ b_{20} = \int_0^1 \frac{x - x_{21}}{x_{20} - x_{21}} \frac{dx}{\sqrt{x}} = 1 + \frac{1}{18} \sqrt{30} $$

$$ b_{21} = \int_0^1 \frac{x - x_{20}}{x_{21} - x_{20}} \frac{dx}{\sqrt{x}} = 1 - \frac{1}{18} \sqrt{30} $$

It can be verified (a computer algebra system like Maple is handy) that this formula also integrates $x^2$ and $x^3$ exactly:

$$ b_{20}x_{20}^2 + b_{21}x_{21}^2 = \frac{2}{5} = \int_0^1 \frac{x^2 \, dx}{\sqrt{x}}. $$

$$ b_{20}x_{20}^3 + b_{21}x_{21}^3 = \frac{2}{7} = \int_0^1 \frac{x^3 \, dx}{\sqrt{x}}. $$

3. Application

In conclusion, we demonstrate the approximation of the integral (1) by these Gaussian formulas. The results are

$$ \int_0^1 e^{-x} \frac{dx}{\sqrt{x}} \approx b_{10} e^{-x_{10}} = 2 \cdot e^{-1/3} \approx 1.433062 $$

$$ \int_0^1 e^{-x} \frac{dx}{\sqrt{x}} \approx b_{20} e^{-x_{20}} + b_{21} e^{-x_{21}} \approx 1.493334 $$

compared to the precise value 1.493648.