AN EXAMPLE OF GAUSSIAN QUADRATURE

In this example we work out the one- and two-point Gaussian quadrature formulas for improper integrals of the form

$$\int_0^1 f(x)\frac{dx}{\sqrt{x}}.$$

We will apply the derived formulas to approximate the integral

$$\int_0^1 e^{-x}\frac{dx}{\sqrt{x}} = \sqrt{\pi}\,\mathrm{erf}(1) \approx 1.493648. \tag{1}$$

1. Inner product, orthogonal polynomials

The nodes of Gaussian quadrature formulas are the roots of certain orthogonal polynomials, so as a preparatory step we work out the first few orthogonal polynomials.

Incorporate the $x^{-1/2}$ singularity into an inner product for functions on $[0,1]$ by defining

$$\langle f, g\rangle = \int_0^1 f(x)g(x)\frac{dx}{\sqrt{x}}. \tag{2}$$

Next, we derive orthogonal polynomials $Q_0(x)$, $Q_1(x)$, $Q_2(x)$ by applying the Gram-Schmidt algorithm of §6.8. First set

$$Q_0(x) = 1. \tag{3}$$

Next, get Q_1 by subtracting from x its component along Q_0:

$$Q_1(x) = x - \frac{\langle xQ_0\rangle}{\langle Q_0Q_0\rangle}Q_0 = x - \frac{2/3}{2}\cdot 1 = x - 1/3. \tag{4}$$

Finally, Q_2 is obtained similarly from x^2:

$$\begin{aligned} Q_2(x) &= x^2 - \frac{\langle x^2Q_0\rangle}{\langle Q_0Q_0\rangle}Q_0(x) - \frac{\langle x^2Q_1\rangle}{\langle Q_1Q_1\rangle}Q_1(x) \\ &= x^2 - \frac{2/5}{2}\cdot 1 - \frac{16/105}{8/45}\cdot(x - 1/3) \\ &= x^2 - \frac{6}{7}x + \frac{3}{35}. \qquad (5) \end{aligned}$$

2. Quadrature Nodes and Weights

2.1. **One-point formula.** The one-point quadrature formula uses for node x_{10} the sole root of $Q_1(x) = x - \frac{1}{3}$, so $x_{10} = \frac{1}{3}$. The weight b_{10} can be found by the method of undetermined coefficients. The weight must be chosen so that the formula integrates the function 1 exactly:

$$b_{10} \cdot 1 = \int_0^1 1 \frac{dx}{\sqrt{x}} = 2. \tag{6}$$

This Gaussian formula integrates the function x exactly too:

$$b_{10} \cdot x_{10} = 2 \cdot \frac{1}{3} = \int_0^1 x \frac{dx}{\sqrt{x}}.$$

2.2. **Two-point formula.** The two-point quadrature formula uses for nodes the roots of $Q_2(x)$ from (5):

$$x_{20} = \frac{3}{7} - \frac{2}{35}\sqrt{30}, \qquad x_{20} = \frac{3}{7} + \frac{2}{35}\sqrt{30}. \tag{7}$$

The weights can again be found by the method of undetermined coefficients, since the formula must integrate the functions 1 and x exactly. Alternatively, we compute them from the Lagrange unit polynomials.

$$b_{20} = \int_0^1 \frac{x - x_{21}}{x_{20} - x_{21}} \frac{dx}{\sqrt{x}} = 1 + \frac{1}{18}\sqrt{30} \tag{8}$$

$$b_{21} = \int_0^1 \frac{x - x_{20}}{x_{21} - x_{20}} \frac{dx}{\sqrt{x}} = 1 - \frac{1}{18}\sqrt{30} \tag{9}$$

It can be verified (a computer algebra system like Maple is handy) that this formula also integrates x^2 and x^3 exactly:

$$b_{20}x_{20}^2 + b_{21}x_{21}^2 = \frac{2}{5} = \int_0^1 x^2 \frac{dx}{\sqrt{x}}.$$

$$b_{20}x_{20}^3 + b_{21}x_{21}^3 = \frac{2}{7} = \int_0^1 x^3 \frac{dx}{\sqrt{x}}.$$

3. Application

In conclusion, we demonstrate the approximation of the integral (1) by these Gaussian formulas. The results are

$$\int_0^1 e^{-x} \frac{dx}{\sqrt{x}} \approx b_{10}e^{-x_{10}} = 2 \cdot e^{-1/3} \approx 1.433062$$

$$\int_0^1 e^{-x} \frac{dx}{\sqrt{x}} \approx b_{20}e^{-x_{20}} + b_{21}e^{-x_{21}} \approx 1.493334$$

compared to the precise value 1.493648.