Math 471 Final Exam

Carry out the *solution* of each problem: show steps of any required calculations; state reasons that justify any conclusions. Mere oracular *answers* will receive no credit.

1. Use a mesh size $h = 1/3$ to set up the difference approximation to Laplace's equation

$$u_{xx} + u_{yy} = 0$$

in the square $\Omega = \{(x, y) : 0 < x, y < 1\}$ with boundary condition

$$u(x, y) = 9x^2 + 18xy - 9y^2$$

on the boundary $\partial\Omega$. Determine the symmetric, positive-definite matrix A and right-side vector b to write the difference equations in the form $Av = b$.

2. Carry out one step each of the Jacobi, Gauss-Seidel, and Conjugate Gradient iterations on the system given in **Computer Problems 4.7, #2**, but use the starting guess $x^{(0)} = (1, -1, 1, -1, 1)^T$.

[Exam Continues on Reverse]

3. On pages 301–303 Kincaid & Cheney derive a QR-decomposition of a certain 4×3 matrix.

(a) Express the orthogonal factor Q and its inverse as matrices whose entries are ratios of integers smaller than 25.

(b) Apply the QR-decomposition to solve the least squares problem

$$\min_x \|Ax - b\|_2$$

with $b = (-126, 63, 21 - 105)^T$. (Feel free to check the solution by any method you like, but demonstrate that you know how to use the QR-decomposition.)

(c) What is the norm of the minimum residual?

4.Toward solving by Newton's method the system of equations

$$\begin{aligned} x - \frac{xy}{20} + 1 &= 0 \\ -2y + \frac{xy}{20} - 1/5 &= 0 \end{aligned}$$

do the following:

(a) Find the Jacobian of the system at a general point (x, y).

(b) Compute the first Newton correction to the initial guess $(x, y) = (40, 20)$ for the solution.

[Exam Continues on Obverse]