

1. Complete the definitions:

(a) (5 points) “An $n \times n$ matrix A is *invertible* if ...’
... there exists an $n \times n$ matrix X such that $AX = XA = I$.”

(b) (5 points) “The *orthogonal complement* of a subspace $V \subset \mathbb{R}^n$ is ...
... the set of all vectors that are orthogonal to every vector in V .”

2. (20 points) Suppose A is an invertible matrix and A^{-1} is known.

Suppose B is obtained from A by switching two columns. Tell how to find B^{-1} from A^{-1} , and prove that your method is correct.

(One way to prove it is to use the fact that if E is an elementary matrix, so that EA is the result of performing one row operation on A , then AE is the result of performing the same operation on the *columns* of A).

Solution: If, say, columns i and j were switched in A to produce B , then we can produce B^{-1} by switching *rows* i and j in A^{-1} .

Proof: Let E be the matrix derived by switching rows i and j in the identity matrix. Notice that $E^{-1} = E$ —we switch rows back just by switching them again.

The hint tells us that $B = AE$. By the Theorem on the Inverse of a Product we have $B^{-1} = E^{-1}A^{-1} = EA^{-1}$; so B^{-1} is the result of switching rows i and j in A^{-1} .

3. (5 points) Complete the definition: “The (indexed) set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is called *linearly independent* if ...
... whenever $c_1\mathbf{v}_1 + \dots + c_k\mathbf{v}_k = \mathbf{0}$, all coefficients $c_1 = \dots = c_k = 0$.

4. (15 points) Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ form a linearly independent set. Prove that the set $\{\mathbf{v}_1, \dots, \mathbf{v}_{k-1}\}$ is linearly independent as well.

Proof: Suppose

$$c_1\mathbf{v}_1 + \dots + c_{k-1}\mathbf{v}_{k-1} = \mathbf{0}. \quad (1)$$

We must show that all the coefficients are zero. To this end, add the equation $0 \cdot \mathbf{v}_k = \mathbf{0}$ to Equation (1): the result is

$$c_1\mathbf{v}_1 + \dots + c_{k-1}\mathbf{v}_{k-1} + 0 \cdot \mathbf{v}_k = \mathbf{0}.$$

Since $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ are linearly independent, *all* the coefficients in this linear relation must be zero, and the proof is complete.

5. Let A be an $m \times n$ matrix. How much can you conclude about the *shape* or the *rank* (or both) of A if ...

- (a) (10 points) The equation $A\mathbf{x} = \mathbf{0}$ has a solution $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$.

Solution: First, A must have exactly three columns ($n = 3$) to be able to multiply a vector in \mathbb{R}^3 . The number of *rows* remains unrestricted.

Second, A has a nontrivial nullspace because the homogeneous equation has a non-trivial solution. So $\text{rank}(A) \leq 2$.

In any case $\text{rank}(A) \leq m$, so the rank of A must lie between zero and the smaller of 2 and m .

- (b) (5 points) The equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

Solution: Every homogeneous system has the trivial solution. This condition does not restrict A , so A can be *any* matrix.

6. (5 points) Let A be an $m \times n$ matrix. Name and define all of the *four fundamental subspaces* associated to A .

Solution: The four fundamental subspaces associated to A are

- (a) The *nullspace*, the set of solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.
 - (b) The *column space*, the span of the column vectors of A .
 - (c) The *row space*, the span of the row vectors of A .
 - (d) The *left nullspace*, the nullspace of A^\top .
7. A certain 4×5 matrix A is brought to its reduced echelon form R by a matrix E that is a product of elementary matrices: that is, $EA = R$, with

$$E = \begin{bmatrix} -1 & 1 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -2 & 1 & -2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 & 1 \\ 0 & 2 & 2 & 2 & 0 \\ -1 & 1 & -1 & 0 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 2 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Find

- (a) (10 points) The rank of A and the dimensions of its four fundamental subspaces;
Solution: Matrix R has three pivots, in columns 1, 2 and 4, so A has rank 3. It follows that the column space and the row space of A have dimension 3.
 By the rank-nullity Theorem (or by counting free variables), the nullspace of A has dimension 2.
 Finally, the left nullspace has dimension (# of rows) minus (rank), or $4 - 3 = 1$.
- (b) (20 points) A basis for each fundamental subspace.

Solution:

Row space The nonzero row vectors of R .

Column space The column vectors 1, 2 and 4 of A , corresponding to pivots in R .

Nullspace $A\mathbf{x} = \mathbf{0}$ iff $R\mathbf{x} = \mathbf{0}$ iff

$$\begin{array}{rcl} x_1 = & -2x_3 & -2x_5 \\ x_2 = & -x_3 & -x_5 \\ x_3 = & x_3 & \\ x_4 = & & x_5 \\ x_5 = & & x_5 \end{array}$$

so a basis for $N(A)$ is $\{[-2, -1, 1, 0, 0]^\top, [-2, -1, 0, 1, 1]^\top\}$

Left nullspace The last row of E .