Math 317 §A Winter 2012 Exam 1 — Sample Solutions

For full credit show the complete solution of each problem including steps of calculations. Support answers by citing definitions and theorems. No credit is allowed for mere answers unsupported by calculations or reasons.

1. Complete the definitions. Let $\mathbf{v}_1, \cdots, \mathbf{v}_k \in \mathbb{R}^n$.
   (a) (5 points) "A **linear combination** of $\mathbf{v}_1, \cdots, \mathbf{v}_k$ is ..."
   a sum of scalar multiples of $\mathbf{v}_1, \cdots, \mathbf{v}_k$.
   (b) (5 points) "The **span** of $\mathbf{v}_1, \cdots, \mathbf{v}_k$ is ..."
   the set of all linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_k$.

2. (20 points) The equations $x_1 = 0$ and $x_2 = 0$ describe planes in $\mathbb{R}^3$. Set up and solve a system of linear equations to find all points that lie on both planes. Express the intersection of these planes as the span of a set of vectors.

   Solution: Points on the intersection of the planes satisfy both equations. Put the equations into matrix form as

   $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

   There is no need to write the augmenting column because the equations are homogeneous.

   The matrix is in reduced echelon form. We read off $x_1 = 0$, $x_2 = 0$, and x_3 is free. The solution set is all vectors of the form $(0, 0, x_3)$, that is, $\text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$.

3. (20 points) Find the angle between the long diagonal of a cube and the diagonal of a face.
HINT: Set up the cube in $\mathbb{R}^3$ with a vertex at the origin, and edges of unit length along the coordinate axes.

Solution: The 8 vertices of the cube are the points (a, b, c) with each of a, b and c taking values 0 or 1. A long diagonal is the one from $(0, 0, 0)$ to $(1, 1, 1)$. Choose the diagonal on the face in the x_1x_2-plane from $(0, 0, 0)$ to $(1, 1, 0)$. The angle θ between these diagonals is given by

$$\begin{aligned}\cos\theta &= \frac{(1,1,1)\cdot(1,1,0)}{\|(1,1,1)\|\;\|(1,1,0)\|}\\ &= \frac{2}{\sqrt{3}\sqrt{2}}\\ &= \frac{1}{3}\sqrt{6}.\end{aligned}$$

4. (20 points) Find a Cartesian equation of the hyperplane through the origin in $\mathbb{R}^4$ spanned by $(1, 0, -1, 0)$, $(0, -1, 0, 1)$ and $(1, 0, 0, -1)$.

Solution: The required equation has the form $\mathbf{a}\cdot\mathbf{x} = 0$; the normal vector $\mathbf{a}$, orthogonal to the vectors spanning the hyperplane, is the solution of the homogeneous system

$$\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & -1 & 0 & 1\\ 1 & 0 & 0 & -1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1\end{bmatrix} \rightsquigarrow \begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 1 & 0 & -1\\ 0 & 0 & 1 & -1\end{bmatrix}$$

The solution is $a_1 = a_4$, $a_2 = a_4$, $a_3 = a_4$ and a_4 is free. Choosing $a_4 = 1$ gives the normal vector $\mathbf{a} = (1, 1, 1, 1)$, so the equation of the hyperplane is

$$x_1 + x_2 + x_3 + x_4 = 0.$$

5. Complete the definitions.

   (a) (5 points) "A square $n \times n$ matrix A is **nonsingular** if ..."
   the rank of A is n.

   (b) (5 points) "If $\mathbf{a} \in \mathbb{R}^n$ is nonzero, the **projection onto a** is ..."
   the function taking a vector $\mathbf{x}$ to

$$\text{proj}_{\mathbf{a}}\mathbf{x} = \frac{\mathbf{a} \cdot \mathbf{x}}{\|\mathbf{a}\|^2}\,\mathbf{a}.$$

6. (20 points) For $\mathbf{a} = (5, -12)$, find the standard matrix of $\text{proj}_{\mathbf{a}}$.

   Solution: The standard matrix of $\text{proj}_{\mathbf{a}}$ is the matrix

$$\begin{bmatrix} \text{proj}_{\mathbf{a}}\mathbf{e}_1 & \text{proj}_{\mathbf{a}}\mathbf{e}_2 \end{bmatrix}$$

   with $\mathbf{e}_1$ and $\mathbf{e}_2$ the standard unit vectors of $\mathbb{R}^2$. Compute

$$\begin{aligned} \text{proj}_{\mathbf{a}}\mathbf{e}_1 &= \frac{5}{13^2}\mathbf{a}, \\ \text{proj}_{\mathbf{a}}\mathbf{e}_2 &= \frac{-12}{13^2}\mathbf{a}, \end{aligned}$$

   so the required matrix is

$$\frac{1}{13^2}\begin{bmatrix} 25 & -60 \\ -60 & 144 \end{bmatrix}.$$