MATH 317: REAL 2×2 MATRICES, COMPLEX EIGENVALUES

1. An Example

(1) Find the (complex) eigenvalues of the matrix $\mathbf{A} = \begin{bmatrix} 0 & -9 \\ 1 & 0 \end{bmatrix}$. [To shorten the work, observe that if $\mathbf{x}$ is an eigenvector belonging to the eigenvalue λ, then $\bar{\mathbf{x}}$ is an eigenvector that belongs to $\bar{\lambda}$.]

(2) Write the eigenvector $\mathbf{x}$ in terms of its real and imaginary parts as

$$\mathbf{x} = \mathbf{u} + i\,\mathbf{v}.$$

(that is, $\mathbf{u}$ and $\mathbf{v}$ are vectors with real components). Form the matrix $\mathbf{P} = [\mathbf{u}\ \mathbf{v}]$ and verify that $\mathbf{P}$ is nonsingular.

(3) Calculate the similar matrix $\mathbf{F} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$. How do the entries of $\mathbf{F}$ compare to the real and imaginary parts of the eigenvalue λ of $\mathbf{A}$?

2. Theorems

Let $\mathbf{A}$ be a 2×2 real matrix. Assume that

$$\lambda = a + bi \qquad a, b \in \mathbb{R}, \qquad b \neq 0$$

is a complex eigenvalue of $\mathbf{A}$. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and assume that the complex vector $\mathbf{x} = \mathbf{u} + i\,\mathbf{v}$ is an eigenvector belonging to λ.

(1) Prove: the vector $\bar{\mathbf{x}}$ is an eigenvector belonging to $\bar{\lambda}$. [Hint: Complex-conjugate the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$.]

(2) Prove:

$$\mathbf{A}\mathbf{u} = a\mathbf{u} - b\mathbf{v}$$
$$\mathbf{A}\mathbf{v} = b\mathbf{u} + a\mathbf{v}$$

[Hint: Expand the equation $\mathbf{A}(\mathbf{u} + i\mathbf{v}) = (a + bi)(\mathbf{u} + i\mathbf{v})$ and collect real and imaginary parts.]

(3) Prove: The matrix $\mathbf{P} = [\mathbf{u}\ \mathbf{v}]$ is nonsingular.

(4) Prove: $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.