

Math 307 §D Midterm Exam II Solutions

1. (10 points) Find $\det(B^5)$ for the matrix

$$B = \begin{bmatrix} 2 & 0 & 0 \\ 30707 & 1 & 0 \\ 4096 & 8192 & 2 \end{bmatrix}$$

Solution: The determinant of a triangular matrix is the product of its diagonal entries, so $\det B = 4$. Since the determinant of a product is the product of the determinants of the factors, $\det(B^5) = (\det B)^5 = 4^5$.

2. (30 points) Define *null space*, *column space*, and *basis*.

Find a basis for $\text{Nul } A$, a basis for $\text{Col } A$ and a basis for $\text{Row } A$.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 4 & 5 & -2 & 6 & 0 \end{bmatrix}$$

Solution: The *null space* of a matrix A is the set of all solutions to the equation $A\mathbf{x} = \mathbf{0}$,

The *column space* of a matrix is the set of all linear combinations of its columns.

A *basis* for a vector space is a linearly independent set that spans the space.

To find a basis for the null space of A , compute its reduced echelon form. After two row replacements you find

$$\text{ref}(A) = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -2 & 2 & 0 \end{bmatrix}.$$

In the homogeneous equation $A\mathbf{x} = \mathbf{0}$, then, x_3, x_4, x_5 are free variables, $x_1 = -2x_3 + x_4$ and $x_2 = 2x_3 - 2x_4$. The solution in parametric form is

$$\mathbf{x} = \begin{bmatrix} -2x_3 + x_4 \\ 2x_3 - 2x_4 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

so a basis for the null space of A is

$$\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

A basis for the column space of A is composed of the columns of A that have pivots:

$$\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$$

Finally, since each row of A has a pivot, the rows of A are linearly independent, so they are a basis for the row space of A .

3. (40 points) Define *linearly dependent* and *rank*.

Answer the following for the matrix

$$A = \begin{bmatrix} 1 & 4 & -3 \\ 2 & 9 & -8 \end{bmatrix}$$

- (a) Are the columns of A linearly dependent?
- (b) Find the rank of A .
- (c) Find a matrix D so that $AD = I_2$, the 2×2 identity matrix.
- (d) Is A invertible?

Solution: A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is *linearly dependent* if there are weights c_1, \dots, c_k not all zero so that $c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k = \mathbf{0}$.

The *rank* of a matrix is the dimension of its column space.

To answer parts (a) to (d), it is useful to row reduce A augmented by the columns of the identity matrix:

$$\begin{bmatrix} 1 & 4 & -3 & 1 & 0 \\ 2 & 9 & -8 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & -3 & 1 & 0 \\ 0 & 1 & -2 & -2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 5 & 9 & -4 \\ 0 & 1 & -2 & -2 & 1 \end{bmatrix}$$

Parts (a) to (d) are now easily answered as follows.

- (a) Yes: because there is no pivot in column three.
- (b) The rank of A is equal to the number of pivots: two.
- (c) There are infinitely many solutions D because x_3 is a free variable. For the simplest solution take $x_3 = 0$ and read off the other variables from the reduced echelon form:

$$D = \begin{bmatrix} 9 & -4 \\ -2 & 1 \\ 0 & 0 \end{bmatrix}$$

- (d) No: a matrix cannot be invertible unless it is square.

4. (20 points) Suppose \mathbf{u} and \mathbf{v} are linearly independent vectors in \mathbb{R}^2 . Explain using determinants why all parallelograms having vertices at $\mathbf{0}$, \mathbf{u} and $\mathbf{v} + t\mathbf{u}$ (with t in \mathbb{R} arbitrary) have the same area.

Solution: The parallelogram with vertices $\mathbf{0}$, \mathbf{u} , \mathbf{v} has (signed) area $\det[\mathbf{u} \ \mathbf{v}]$.

The parallelogram with vertices $\mathbf{0}$, \mathbf{u} , $\mathbf{v} + t\mathbf{u}$ has (signed) area $\det[\mathbf{u} \ (\mathbf{v} + t\mathbf{u})]$. In this matrix, subtract t times column one from column two. Since column replacement leaves the determinant unchanged we have

$$\det[\mathbf{u} \ (\mathbf{v} + t\mathbf{u})] = \det[\mathbf{u} \ \mathbf{v}]$$

so all these parallelograms have the same area.