Math 301 §A 3rd Midterm Exam — Sample Solutions

For full credit show the complete solution of each problem including steps of calculations. Support answers by citing definitions and theorems. No credit will be given for mere answers unsupported by calculations or reasons.

1. (5 points) Complete the definition: "A *homomorphism* f from a group G to a group P is a function $f : G \to P$ such that for all $x, y \in G$, $f(xy) = f(x)f(y)$."

2. (5 points) Complete the definition: "The *kernel* of a homomorphism $f : G \to P$ is $f^{-1}\{e\}$."

3. (20 points) Let $f : G \to P$ be a group homomorphism. Suppose K is a normal subgroup of P. Prove that

$$f^{-1}(K) = \{g \in G : f(g) \in K\}$$

is a normal subgroup of G.

Solution: See the proof of items 7 and 8 of Theorem 10.2, page 203 of the textbook.

4. (5 points) Complete the definition: "A subgroup H of a group G is called a *normal* subgroup of G if every left coset of H is a right coset of H." [Alternatively: "if for every $g \in G$, $gHg^{-1} = H$."

5. (5 points) Complete the definition: "We say that G is the *internal direct product* of H and K and write $G = H \times K$ if H and K are normal subgroups of G, $HK = G$, and $H \cap K = \{e\}$."

6. (20 points) Are the groups $U(55)$ and $U(75)$ in the same isomorphism class?

   Solution: We have

$$\begin{aligned} U(55) &= U(5) \oplus U(11) \\ &= \mathbf{Z}_4 \oplus \mathbf{Z}_{10} \\ &= \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_5. \end{aligned}$$

   On the other hand,

$$\begin{aligned} U(75) &= U(3) \oplus U(25) \\ &= \mathbf{Z}_2 \oplus \mathbf{Z}_{20} \\ &= \mathbf{Z}_2 \oplus \mathbf{Z}_4 \oplus \mathbf{Z}_5, \end{aligned}$$

   so the two groups *are* in the same isomorphism class.

7. (20 points) Find all elements of order 2 in $Z_2 \oplus Z_2 \oplus Z_2$.

   Solution: All seven nonzero elements in the group have order two: $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, and $(1,1,1)$.

8. (20 points) Let G be a finite group, and H a normal subgroup of G. For any element $g \in G$, let $|g|$ denote the order of g in G and $|gH|$ the order of gH in the factor group G/H.

   Prove that $|gH|$ divides $|g|$.

   Proof: Let k be the order of g in G, so $g^k = e$. Then in the group G/H we get $(gH)^k = g^k H = eH = H$, so the order of gH in G/H is a divisor of k.

   Alternative proof: Let $\pi : G \to G/H$ be the canonical projection defined by $\pi(g) = gH$. Then π is a homomorphism and the desired result follows by part 3 of Theorem 10.1 of the textbook.