Math 301 §A 2nd Midterm Exam — Sample Solutions

For full credit show the complete solution of each problem including steps of calculations. Support answers by citing definitions and theorems. No credit will be given for mere answers unsupported by calculations or reasons.

1. (20 points) Let a be an element of order 48 in a group. Find k, a divisor of 48, so that $\langle a^{21} \rangle = \langle a^k \rangle$.

   Solution: In a cyclic group $\langle a \rangle$ of order n, the subgroup $\langle a^j \rangle$ is the same as the subgroup $\langle a^k \rangle$ if $k = \text{GCD}(n, j)$.

   Use Euclid's algorithm to compute GCD(48, 21):

$$\begin{aligned} 48 &= 2 \cdot 21 + 6 \\ 21 &= 3 \cdot 6 + 3 \\ 6 &= 2 \cdot 3. \end{aligned}$$

   So the $\text{GCD}(48, 21) = 3$; therefore $\langle a^{21} \rangle = \langle a^3 \rangle$.

   As a check, it's obvious since $a^{21} = (a^3)^7$ that $\langle a^{21} \rangle \subset \langle a^3 \rangle$. For the converse, back-tracking in the GCD calculation above gives $3 = 7 \cdot 21 - 3 \cdot 48$, so $a^3 = (a^{21})^7$ and so $\langle a^3 \rangle \subset \langle a^{21} \rangle$.

2. (20 points) Let $\beta \in S_7$. If $\beta^4 = (2143567)$, express β as a product of disjoint cycles.

   Solution: Since β^4 is a 7-cycle in S_7, so is β; hence $\beta^7 = e$ and

$$\beta = \beta^8 = (\beta^4)^2 = (2457136).$$

3. (5 points) Complete the definition: "An *automorphism* of a group G is ..."

   an isomorphism of G with itself.

4. (5 points) Complete the definition: "The *center*, $Z(G)$, of a group G is ..."

   the set of elements of G that commute with every element of G.

5. (20 points) Let G be a group, and let $g, h \in G$. Prove: if $gxg^{-1} = hxh^{-1}$ for every $x \in G$, then $h^{-1}g \in Z(G)$.

   Solution: Assume that

   $$(\forall x \in G)\ gxg^{-1} = hxh^{-1}.$$

   Left-multiplying by h^{-1} and right-multiplying by g leads to

   $$(\forall x \in G)\ (h^{-1}g)x = x(h^{-1}g).$$

   This means that $h^{-1}g$ commutes with every element of G; that is, $h^{-1}g \in Z(G)$.

6. (5 points) Complete the definition: "Let G be a group, and let H be a subgroup of G. For any $a \in G$ the *left coset of H containing a* is ...

   the set $aH = \{ah : h \in H\}$.

7. (5 points) Complete the definition: "Let G be a group of permutations of a set S. For each $s \in S$ the *orbit of s under G* is ..."

   the subset $\text{orb}_G(s) = \{g \cdot s : g \in G\} \subset S$.

8. Let A, B, C, D be the vertices of a regular tetrahdron. The group of rotations of the tetrahedron is a permutation group on the set of edges $\{AB, AC, AD, BC, BD, CD\}$. (Recall that the rotation group is isomorphic to A_4.)

   (a) (10 points) Find the orbit of AB.

   Solution:

   (1) The identity rotation takes AB to AB.

   (2) The rotations about the axis through A and the center of its opposite face take AB to AC and AD.

   (3) The rotations about the axis through B and the center of its opposite face take AB to BC and BD.

   (4) The 180° rotation about the axis joining the midpoint of AC to the midpoint of BD takes AB to CD.

   Therefore the orbit of AB is the whole set of six edges of the tetrahedron.

   (b) (10 points) Find the stabilizer of AB. Confirm that the orbit-stabilizer theorem is satisfied in the case of AB.

   Solution: Just two rotations of the tetrahedron leave the edge AB fixed:

   (1) The identity rotation; and

   (2) The 180° rotation about the axis joining the midpoint of AB to the midpoint of the opposite edge CD.

   The orbit-stabilizer theorem declares that "the order of the orbit is the index of the stabilizer."

   The order of the orbit of AB is 6, the number of edges in the orbit of AB.

   The index of the stabilizer of AB is the number of cosets of its 2-element stabilizer subgroup:

$$[A_4 : \text{stab}_{A_4}(AB)] = |A_4|/|\text{stab}_{A_4}(AB)| = 12/2 = 6,$$

   confirming the orbit-stabilizer theorem.