

## Mathematics 273 Problem Set 5

Due Friday, December 14, 2001

**By submitting a paper for this assignment you declare that you are not submitting any unattributed work of any other person.**

Hand in all analytical work that contributes to your solution. Submit printed copies of all MATLAB scripts (except M-files provided with the textbook), output tables, and graphs. Use the `diary` feature to get printed output from MATLAB sessions. Be sure to

- (a) Exploit vectorization whenever possible.
- (b) Include explanatory comments in all programs.
- (c) Make informative labels for all tables and graphs.
- (d) Edit output to make it compact and easy to read.

1. Suppose there are  $n$  surplus electrons on a metallic sphere. The sphere conducts electricity, so the electrons are free to move around; they distribute themselves so as to minimize the total electrostatic potential. (The electrostatic potential between any pair of electrons is proportional to the reciprocal of the distance between them. The total potential is the sum of the potentials of all pairs.) Find an equilibrium distribution by minimizing the total potential.

Assume a sphere of unit radius, and use spherical coordinates  $(\theta_i, \varphi_i)$  for the position of the  $i$ th electron. Coordinate  $0 \leq \theta \leq 2\pi$  is the longitude, and  $0 \leq \varphi \leq \pi$  is the colatitude, i.e.  $\varphi \rightarrow 0$  approaches the north pole and  $\varphi \rightarrow \pi$  approaches the south pole of the sphere.

Remove the degeneracy of the problem due to spherical symmetry by putting the first electron at the north pole ( $\theta_1 = 0 = \varphi_1$ ), the second electron on the zeroth meridian ( $\theta_2 = 0$ ). The two-electron problem has one design variable,  $\varphi_2$ ; the minimum is achieved by placing the second electron at the south pole.

- (a) Use Steepest Descent to find the equilibrium distribution of four electrons (a five-variable problem: design variables are  $(\varphi_2, \theta_3, \varphi_3, \theta_4, \varphi_4)$ ). Modify the text's `ShowSD` script as necessary, and choose the length of each Steepest Descent step by a mouse click. [To compute gradients of the potential, use analytical partial derivatives or divided differences as convenient.]
- (b) Use the MATLAB function `fmins` to find the equilibrium distribution for six electrons (a nine-variable problem).

For the initial configuration in each case, put electrons `2:n` around the equator at equally spaced longitudes, and random small colatitudes  $\frac{\pi}{2} \pm \Delta\varphi$ .

Describe each solution in English. (It helps to divide the solution angles by  $\pi$  so they can be recognized, if possible, as simple fractions.)

2. Use the *shooting method* to solve the second-order linear differential equation

$$y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0 \quad 1 < x < 6 \quad (1)$$

with the boundary conditions  $y(1) = 1$ ,  $y(6) = 0$ , as follows.

First, use the MATLAB ODE solver of your choice to compute  $Y_1$ , the solution of (1) satisfying the initial conditions  $Y_1(1) = 1$ ,  $Y_1'(1) = 0$ , and  $Y_2$ , the solution satisfying  $Y_2(1) = 0$ ,  $Y_2'(1) = 1$ , both on the interval  $1 \leq x \leq 6$ .

Next, explain why the solution can be expressed in the form  $y = C_1 Y_1 + C_2 Y_2$  with some constants  $C_1$  and  $C_2$ . Explain why  $C_1 = 1$ . Finally, since  $y(6) = 0 = Y_1(6) + C_2 Y_2(6)$ , it follows that  $C_2 = -Y_1(6)/Y_2(6)$ .

Compute and display  $C_2$ , and plot  $y$  to show that it takes on the required boundary values.

**3.** Fig. 1 shows a circuit diagram of a transistor amplifier. The operating voltage is  $U_b = 6$ , the entry voltage is  $U_e(t)$ ,  $U_i(t) (i = 1 : 5)$  are the voltages at the nodes 1 : 5, and  $U_5(t)$  is the output voltage. The transistor acts as an amplifier in that the current from node 4 to node 3 is 99 times as large as that from node 2 to node 3, and depends on the voltage difference  $U_3 - U_2$  in a nonlinear way. Kirchhoff's law says that the sum of currents entering a node vanishes. This law applied to the 5 nodes of Fig. 1 leads to the following equations:

$$\begin{aligned} \text{node 1:} \quad & \frac{U_e(t)}{R_0} - \frac{U_1}{R_0} + C_1(U_2' - U_1') = 0 \\ \text{node 2:} \quad & \frac{U_b}{R_2} - U_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + C_1(U_1' - U_2') - 0.01f(U_2 - U_3) = 0 \\ \text{node 3:} \quad & f(U_2 - U_3) - \frac{U_3}{R_3} - C_2U_3' = 0 \\ \text{node 4:} \quad & \frac{U_b}{R_4} - \frac{U_4}{R_4} + C_3(U_5' - U_4') - 0.99f(U_2 - U_3) = 0 \\ \text{node 5:} \quad & -\frac{U_5}{R_5} + C_3(U_4' - U_5') = 0 \end{aligned}$$

The constants and the transistor characteristic are given by

$$\begin{aligned} R_0 &= 1000, \quad R_1 = \dots = R_5 = 9000 \\ C_k &= k \cdot 10^{-6}, \quad k = 1 : 3 \\ f(U) &= 10^{-6} \left( \exp\left(\frac{U}{0.026}\right) - 1 \right) \end{aligned}$$

and the input signal is  $U_e(t) = 0.4 \cdot \sin(200\pi t)$ . Put the system into the form  $MU' = F(t, U)$  with nodal voltage vector  $U = (U_1, \dots, U_5)^T$  and *mass matrix*

$$M = \begin{bmatrix} -C_1 & C_1 & & & \\ C_1 & -C_1 & & & \\ & & -C_2 & & \\ & & & -C_3 & C_3 \\ & & & C_3 & -C_3 \end{bmatrix}$$

Use `ode15s` to solve this system for  $0 \leq t \leq 0.20$  with the initial values

$$U_1(0) = 0, \quad U_2(0) = U_3(0) = \frac{U_b R_1}{R_1 + R_2}, \quad U_4(0) = U_b, \quad U_5(0) = 0.$$

Plot the input voltage  $U_e(t)$  together with the output voltage  $U_5(t)$ .

