Math 273 Midterm I Solutions

1. The fragment on the left computes the dot product of the vectors x and y. In MATLAB do:

```
>> p = x'*y
```

The right-hand fragment computes the outer product of the vectors x and y. In MATLAB do

```
>> a = y*x';
```

2. A quadratic equation has non-real roots when its discriminant $B^2 - 4AC$ is negative; the displayed code counts the number of times this occurs among the 1000 random samples. Vectorize the count in MATLAB by

```
>> ncomplex = sum( B.^2 - 4*A.*C < 0 );
```

3. Calculate ($h \neq 0$ is arbitrary)

$$
\begin{aligned}
C_h(a) &= \frac{y(a+h) - y(a-h)}{2h} \\
&= \frac{A(a+h)^2 + B(a+h) + C - \big(A(a-h)^2 + B(a-h) + C\big)}{2h} \\
&= \frac{2Aah + Bh + 2Aah + Bh}{2h} = 2Aa + B = y'(a)
\end{aligned}
$$

by routine algebraic expansion and simplification.

In the homework we learned that $|C_h - y'(a)| \leq M_3 h^2/6$ provided M_3 is a bound for y''' near a (pp. 57–58). Since y is a quadratic function, $y''' = 0$ everywhere, so we can take $M_3 = 0$ and deduce $|C_h - y'(a)| = 0$, that is, the centered difference delivers the exact derivative.

4.

(a) We get the VanderMonde system of equations by substituting the data points into the equation $a_1 + a_2 x + a_3 x^2 = y$. In matrix form, the equations are:

$$
\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
$$

(b) This system can be solved by (for example) forming the augmented matrix and performing elimination. First subtract equation 1 from equation 2 and equation 3:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 2 & 8 & 1 \end{array}\right]$$

Then subtract 2 times equation 2 from equation 3:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right]$$

Reading from bottom to top we find

$$\begin{aligned} 2a_3 &= 1, \quad \text{so} \quad a_3 = \frac{1}{2} \\ a_2 + 3 \cdot \frac{1}{2} &= 0, \quad \text{so} \quad a_2 = -\frac{3}{2} \\ a_1 + (-\frac{3}{2}) + \frac{1}{2} &= 0, \quad \text{so} \quad a_1 = 1 \end{aligned}$$

(c) For the coefficients of the Newton form of the interpolating polynomial, substitute the data successively into the equation

$$c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2) = y.$$

Substituting $x = 1$, $y = 0$ gives $c_1 = 0$.

Next, substitute $x = 2$, $y = 0$ and use $c_1 = 0$, to find $c_2 = 0$ also.

Finally, substituting $x = 3$, $y = 1$, $c_1 = c_2 = 0$ leads to $2c_3 = 1$, so that $c_3 = \frac{1}{2}$.

5.

(a) If x is close to 1, the term of largest magnitude in the expansion of $p(x) = (x-1)^6$ is $T = -20x^3$, and its magnitude is about 20.

Since IEEE long binary floating point numbers have 53 significant bits, this number system has $\beta = 2$ and $t = 53$. Now, $20 = (.101)_{\mathrm{B}} \times 2^5$, so we expect (p. 46)

$$|\mathrm{fl}(T) - T| \leq \frac{1}{2} 2^{5-53} = 2^{-49} \approx 1.8 \times 10^{-15}.$$

Alternatively, by the rule of thumb used in lecture, we expect a quantity of magnitude 20 to contain roundoff error of about $20\,\texttt{eps} \approx 4 \times 10^{-15}$.

(b) It matches pretty well the scale ($\pm 5 \times 10^{-15}$) in the Figure. The fluctuations in the Figure can be attributed to the random accumulation and cancellation of roundoff errors (of sizes $15\,\texttt{eps}$, $6\,\texttt{eps}$,...) from the other terms.