Math 273 Midterm II Solutions

1. First subtract $\frac{1}{2}\times$ row 1 from row 2 and $\frac{1}{3}\times$ row 1 from row 3 to obtain

$$A = \begin{bmatrix} 60 & 30 & 20 \\ 0 & 5 & 5 \\ 0 & 5 & 5\frac{1}{3} \end{bmatrix}$$

Then subtract row 2 from row 3 to make A upper triangular; collecting the multipliers below the diagonal gives L:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 60 & 30 & 20 \\ 0 & 5 & 5 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

To solve $Ax = b = [0; 0; 1]$, first do forward substitution to solve $Ly = b$. The solution is obviously $y_1 = y_2 = 0$, $y_3 = 1$. Performing back substitution to solve $Ux = y$ gives

$$\begin{aligned} \frac{1}{3}x_3 = 1 &\quad\Rightarrow\quad x_3 = 3 \\ 5x_2 + 5x_3 = 0 &\quad\Rightarrow\quad x_2 = -3 \\ 60x_1 + 30x_2 + 20x_3 = 0 &\quad\Rightarrow\quad x_1 = \frac{1}{2}. \end{aligned}$$

2. For M we have the row and column sums

$$\begin{array}{rcl} 0.1481 + 0.0740 & = & 0.2221 \\ +0.4445 + 0.2221 & = & 0.6666 \\ 0.5926 \quad 0.2961 & & \end{array}$$

The corresponding quantities for M^{-1} are $10^8\times$

$$\begin{array}{rcl} 0.2221 + 0.0740 & = & 0.2961 \\ +0.4445 + 0.1481 & = & 0.5926 \\ 0.6666 \quad 0.2221 & & \end{array}$$

The infinity-norm is the maximum row sum:

$$\|M\|_\infty = 0.6666 \qquad \|M^{-1}\|_\infty = 0.5926 \times 10^8$$

so $\kappa_\infty(M) = 39502716$.

The 1-norm is the maximum column sum:

$$\|M\|_1 = 0.5926 \qquad \|M^{-1}\|_1 = 0.6666 \times 10^8$$

so $\kappa_1(M) = 39502716$ too.

3. `MatVecRO` computes the matrix-vector product `y=A*x` by a row-oriented algorithm using dot products. (The comments are correct about the *result*, but not the *method*.) For each dot product it computes n products of pairs of scalars (n flops) and adds them (n flops more) for a total of $2n$ flops. There are n dot products, making for $2n^2$ flops.

`MatVecCO` computes the same product by a column-oriented algorithm using saxpys. (It is correctly described by the comments.) Each time through the loop a column of A is multiplied by a component of x (n flops) and the result is added to y (n flops). The loop is repeated n times for a total of $n(n+n) = 2n^2$ flops.

4. The matrix A and vector b are

$$A = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \qquad b = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

A rotation matrix G that makes A upper triangular is found by making the second component of GA zero ($s = \sin\theta$, $c = \cos\theta$):

$$G \cdot A = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3c + 4s \\ 3s + 4c \end{bmatrix}.$$

From $3\sin\theta + 4\cos\theta = 0$ follows $\cot\theta = -\frac{3}{4}$; thus $\sin\theta = 1/\sqrt{1+\cot^2\theta} = 4/5$ and $\cos\theta = \sin\theta\cot\theta = -\frac{3}{5}$. Then $A = QR$ with $Q = G^T$ and $R = [5, 0]^T$.

To solve the least squares problem, multiply the residual $Ax - b$ by G, obtaining

$$GAx - Gb = Rx - Gb = \begin{bmatrix} 5 \\ 0 \end{bmatrix} x - \begin{bmatrix} 7 \\ 1 \end{bmatrix}.$$

The least squares solution is $x^* = \frac{7}{5}$; the minimum residual is $Ax^* - b = [4/5; 3/5]$, and its norm is 1.