

Review of Taylor Approximation

If you know a function and some of its derivatives at one point, you can approximate the function at nearby points. The Taylor approximation of degree n of a function f about a point a is

$$\begin{aligned} f(x) &\approx \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k \\ &= f(a) + f'(a)(x - a) + \frac{1}{2}f''(a)(x - a)^2 + \dots \end{aligned}$$

Example: Here is the Taylor approximation of degree 4 of $f(x) = e^{-x}$ about 0. First compute the value of f and its first four derivatives at 0, filling in the following table.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{-x}	1
1	$-e^{-x}$	-1
2	e^{-x}	1
3	$-e^{-x}$	-1
4	e^{-x}	1

Then use the coefficients in the third column to construct the Taylor approximation:

$$e^{-x} \approx 1 + (-1)x + \frac{1}{2}x^2 + \frac{-1}{6}x^3 + \frac{1}{24}x^4$$

Taylor series

When a function has derivatives of all orders, its Taylor approximation may be carried out to any degree. The function is represented by a Taylor series if the resulting series converges to the function. Here are some elementary functions and their Taylor series.

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$\exp x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k \quad |x| < 1$$

$$\log(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k} \quad |x| < 1$$

The Remainder

The discrepancy between a function and its Taylor approximation is called the remainder. The remainder can be represented by a formula attributed to Lagrange: there is a point ξ between x and a so that

$$f(x) - \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k = \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-a)^{n+1}$$

In the previous example of Taylor approximation of degree 4 of e^{-x} the remainder is

$$\frac{-e^{-\xi}}{5!} x^5.$$

The remainder formula is used to estimate or bound the size of the error in the Taylor approximation. In the example, if the Taylor approximation of degree 4 is used for e^{-x} for $-1 \leq x \leq 1$, then the error has magnitude at most

$$\frac{e^{-(-1)}}{120} 1^5 = e/120 \doteq 0.023$$