Math 267 Project 2: Periodically Forced Oscillator

Due Thursday, March 24, 2005

In this project you will use the Laplace transform to find the steady state response of the oscillator with damping and periodic forcing,

$$y'' + 6y' + 13y = f(t) \qquad y(0) = y_0 \qquad y'(0) = v_0. \tag{1}$$

The forcing function $f(t)$ is the 2π-periodic *sawtooth* waveform defined by

$$f(t) = \begin{cases} t & 0 < t \le \pi \\ t - 2\pi & \pi < t \le 2\pi \end{cases} \qquad\qquad f(t + 2\pi) = f(t). \tag{2}$$

1 Periodic Forcing, Poles and Residues

Solutions to Problems 1–2 are due Thursday, March 10.

Look first at a forcing function that is a superposition of two waves, a *fundamental* $\cos t$ and a *harmonic* $\cos 2t$ with double the fundamental frequency:

$$y'' + 6y' + 13y = 30\cos t + 75\cos 2t \tag{3}$$

The steady state can be found by the method of undetermined coefficients.

Problem 1. Find a particular solution of (3) by the method of undetermined coefficients.

The advantage of this method over the Laplace transform method for finding the periodic steady state is apparent. Using undetermined coefficients, one computes the steady state directly. But the Laplace transform method, using partial fractions to find the inverse Laplace transform of $Y(s) = \mathcal{L}(y)(s)$

$$Y(s) = \frac{30s}{(s^2+1)(s^2+6s+13)} + \frac{75s}{(s^2+4)(s^2+6s+13)} + \frac{sy_0 + v_0}{s^2+6s+13} \tag{4}$$

(the initial conditions $y(0) = y_0$, $y'(0) = v_0$ have been used) seems to entail computing both the transient and the steady state.

There is, however, a way to extract just the steady state part of Eq. (4).

First, some vocabulary. A function such as

$$F(s) = \frac{A}{s-a}$$

is said to have a *pole* at $s = a$; the constant A in the numerator is called the *residue* at the pole. A pole at a with residue A contributes a term Ae^{at} to the inverse Laplace transform.

Taking Laplace transforms of the expressions for sine and cosine in terms of complex exponentials $\cos t = \frac{1}{2}\left(e^{it} + e^{-it}\right)$, $\sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right)$ gives the complex partial fractions

$$\frac{s}{s^2+1} = \frac{1/2}{s-i} + \frac{1/2}{s+i} \qquad\qquad \frac{1}{s^2+1} = \frac{-i/2}{s-i} + \frac{i/2}{s+i}.$$

We say that the Laplace transform of the cosine has poles at $s = i$ and at $s = -i$, each with residue $1/2$. The Laplace transform of the sine has the *same* poles but *different* residues $-i/2$ and $i/2$, respectively.

Observe an important fact: when $F(s)$ is real for real s, its poles appear in complex conjugate pairs, and the residues at complex conjugate poles are complex conjugates of each other.[1]

We use this fact to write $Y(s)$ in Eq. (4) in partial fractions with linear denominators (factoring $s^2+6s+13 = (s+3)^2-(-4) = (s+3-2i)(s+3+2i)$)

$$Y(s) = \frac{A}{s-i} + \frac{\bar{A}}{s+i} + \frac{B}{s-2i} + \frac{\bar{B}}{s+2i} + \frac{C}{s-(-3+2i)} + \frac{\bar{C}}{s-(-3-2i)}. \quad (5)$$

The residues A, B and C are still unknown. Now, the periodic part of the solution y comes from the purely imaginary poles at $s = \pm i$ and $s = \pm 2i$. The poles at $s = -3 \pm 2i$ contribute only to the transient. So we need A and B but we do *not* need C. Here is the crucial point: the formula above shows that we can compute the residue A directly by evaluating the limit

$$\begin{aligned}
A &= \lim_{s\to i}(s-i)Y(s) \\
&= \lim_{s\to i}\left[\frac{s-i}{s^2+1}\frac{30s}{s^2+6s+13} + (s-i)\left(\frac{75s}{(s^2+4)(s^2+6s+13)} + \frac{sy_0+v_0}{s^2+6s+13)}\right)\right] \\
&= \lim_{s\to i}\frac{s-i}{s^2+1}\cdot\lim_{s\to i}\frac{30s}{s^2+6s+13} + 0 \quad \text{(Limit of product } = \text{product of limits.)} \\
&= \lim_{s\to i}\frac{1}{2s}\cdot\frac{30i}{-1+6i+13} = \frac{30i}{2i(12+6i)} \quad \text{(Lhospital's Rule in first limit.)} \\
&= \frac{5}{2}\frac{1}{2+i}\cdot\frac{2-i}{2-i} = \frac{5}{2}\frac{2-i}{5} = 1 - \frac{1}{2}i.
\end{aligned}$$

[In the last line we multiply by the complex conjugate to rationalize the denominator.] Thus the part of the particular solution that is the response to $30\cos t$ is the inverse Laplace transform of

$$\frac{A}{s-i} + \frac{\bar{A}}{s+i} = \frac{1-\frac{1}{2}i}{s-i} + \frac{1+\frac{1}{2}i}{s+i} = \frac{2s+1}{s^2+1}$$

or $2\cos t + \sin t$, as you found in the solution to **Problem 1**.

Problem 2. Find the residue B by the same method just used to find A, and verify that the result agrees with the solution to **Problem 1**.

2 Periodic steady state – s-domain

The periodic steady state solution of Eq. (1) could be found by a piecewise application of the method of undetermined coefficients. The computation would

[1] Recall that the *complex conjugate* of a complex number $z = a + bi$ is $\bar{z} = a - bi$, the *real part* is $\operatorname{Re} z = a$ and the *imaginary part* is $\operatorname{Im} z = b$. Thus $z + \bar{z} = 2\operatorname{Re} z$ and $z - \bar{z} = 2i\operatorname{Im} z$.

be long and tedious. Instead, the method of the previous section gives a short, elegant solution.

The Laplace transform of the solution of Eq. (1) is expressed in terms of the Laplace transform $F(s)$ of the forcing function $f(t)$ in Eq. (2) by

$$Y(s) = \frac{F(s)}{s^2 + 6s + 13} + \frac{sy_0 + v_0}{s^2 + 6s + 13}. \tag{6}$$

Problem 3. Find $F(s)$. [Use Proposition 5.17 and Example 5.20 on pages 273–275 as a guide.]

As we saw in the previous section, the poles at $s = -3 \pm 2i$ due to $s^2+6s+13$ in the denominator contribute only to the transient. The periodic steady state comes from the purely imaginary poles of $Y(s)$: these are just the poles of $F(s)$.

Problem 4.

(a) Show that $F(s)$ does *not* have a pole at $s = 0$.

(b) Show that $F(s)$, and hence $Y(s)$, has a pole at $s = \pm ni$ for every $n = 1, 2, 3, \ldots$, and no other purely imaginary poles. [What is $e^{-2n\pi i}$?]

(c) Find the residue $A_n = \lim_{s \to ni}(s - ni)Y(s)$ at each pole $s = ni$.

The result of **Problem 4** is that the Laplace transform of the periodic steady state solution of Eq. (1) can be written as an infinite series

$$Y(s) = \sum_{n=1}^{\infty} \frac{A_n}{s - ni} + \frac{\bar{A}_n}{s + ni} = \sum_{n=1}^{\infty} \frac{2\,\mathrm{Re}\,A_n\, s - 2n\,\mathrm{Im}\,A_n}{s^2 + n^2}. \tag{7}$$

3 Periodic steady state – t-domain

Problem 5. Use the residues A_n found in your solution of **Problem 4** to express the inverse Laplace transform of $Y(s)$ from Eq. (7) in the form

$$y(t) = \sum_{n=1}^{\infty} a_n \cos nt + b_n \sin nt. \tag{8}$$

This expression of a periodic function as a superposition of a fundamental frequency ($\cos t$, $\sin t$) and its harmonics is an example of a *Fourier series.*

Problem 6. Truncate the series (8) and plot the solution. Find approximations to $y(0)$ and $y'(0)$.

4 Alternate Solution for Example 5.21

Here is how the present method applies to the worked Example 5.21 on pages 275–276. The Laplace transform of the solution is

$$Y(s) = \frac{1}{s(s^2 + 1)} \frac{1 - e^{-s}}{1 + e^{-s}}.$$

There is no pole at $s = 0$ because $\lim_{s\to 0} sY(s) = 0$. Obviously there are poles at $s = \pm i$ (there is no damping term). There are also poles at odd multiples of πi, $s = (2n+1)\pi i$, $n = 0, 1, 2, \ldots$ because $e^{-(2n+1)\pi i} = -1$. Hence $Y(s)$ can be represented by

$$Y(s) = \frac{B}{s-i} + \frac{\bar{B}}{s+i} + \sum_{n=0}^{\infty} \frac{A}{s-(2n+1)\pi i} + \frac{\bar{A}}{s+(2n+1)\pi i}$$

Because there is no damping, the solution will be a superposition of a wave at the natural frequency ($\cos t$, $\sin t$) and a wave consisting of the fundamental forcing frequency ($\cos \pi t$, $\sin \pi t$) and its odd harmonics. The residues are computed by

$$\begin{aligned} B &= \lim_{s\to i}(s-i)\frac{1}{s(s^2+1)}\frac{1-e^{-s}}{1+e^{-s}} \\ &= \lim_{s\to i}\frac{s-i}{s^2+1}\cdot\lim_{s\to i}\frac{1}{s}\frac{1-e^{-s}}{1+e^{-s}} \\ &= \frac{1}{2i}\cdot\frac{1}{i}\cdot\frac{1-(\cos 1 - i\sin 1)}{1+(\cos 1 - i\sin 1)} = (-i/2)\tan\frac{1}{2} \end{aligned}$$

after some simplification; and

$$\begin{aligned} A_n &= \lim_{s\to(2n+1)\pi i}(s-(2n+1)\pi i)\frac{1}{s(s^2+1)}\frac{1-e^{-s}}{1+e^{-s}} \\ &= \lim_{s\to(2n+1)\pi i}\frac{s-(2n+1)\pi i}{1+e^{-s}}\cdot\lim_{s\to(2n+1)\pi i}\frac{1-e^{-s}}{s(s^2+1)} \\ &= \frac{2i}{(2n+1)\pi\left[(2n+1)^2\pi^2-1\right]}. \end{aligned}$$

Substituting and simplifying, we find

$$Y(s) = \frac{\tan\frac{1}{2}}{s^2+1} + \sum_{n=0}^{\infty}\frac{-4}{\left[(2n+1)^2\pi^2-1\right]}\frac{1}{s^2+(2n+1)^2\pi^2}$$

from which we can read off the inverse Laplace transform as

$$y(t) = (\tan\frac{1}{2})\sin t - 4\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi t}{(2n+1)\pi\left[(2n+1)^2\pi^2-1\right]}.$$

The series converges absolutely, by comparison with $\sum_{n=0}^{\infty}(2n+1)^{-3}$. It satisfies the initial conditions $y(0) = 0$, $y'(0) = 0$. All truncations of the series obviously satisfy $y(0) = 0$. Taking the terms $n = 0$ and $n = 1$ of the series gives an approximation with $y'(0) \approx .05$; retaining terms through $\sin 9\pi t$ gives $y'(0) \approx .0202$.

Plotting just the first two terms of the series produces a very good replica of Figure 12 on page 276.