Math 267 §B2 Midterm Exam Sample Solutions

Carry out the *solution* of each problem: show steps of any required calculations; state reasons that justify any conclusions. Mere oracular *answers* will receive no credit.

1. Consider the equation for the velocity of a falling body with air resistance,

$$mv' = -mg - rv$$

with positive constants m (mass), g (acceleration of gravity) and r (coefficient of friction in air).

(a) Find the equilibrium solution.

Solution: Since this equation is autonomous the equilibrium solution is the constant solution of $-mg - rv = 0$, that is, $v = -mg/r$.

(b) Is the equilibrium stable?

Solution: Since $\dfrac{d}{dv}(-mg - rv) = -r < 0$, the equilibrium is stable.

(c) What is the physical meaning of the equilibrium?

Solution: This equilibium is the terminal velocity. It is the velocity at which the gravity force $-mg$ is balanced by the air resistance force $-rv$.

2. Find a particular solution of the differential equation

$$y'' + 2y' + 2y = 5\cos t + e^{-t}(6t + 7)$$

by the method of undetermined coefficients.

Solution: By superposition, a particular solution may be written as $y_p = u + v$, with

$$u'' + 2u' + 2u = 5\cos t, \qquad v'' + 2v' + 2v = e^{-t}(6t + 7).$$

The characteristic roots satisfy $r^2 + 2r + 2 = 0$ or $(r + 1)^2 = -1$, so they are $r = -1 \pm i$. Neither i nor -1 is a characteristic root, so the particular solutions have the form $u = \operatorname{Re} Ae^{it}$, $v = e^{-t}(Bt + C)$.

Substituting the form of u into the differential equation gives

$$\begin{aligned}(D^2+2D+2)\cdot Ae^{it} &= (i^2+2i+2)Ae^{it} = 5e^{it}\\ (1+2i)A &= 5\\ A &= \frac{5}{1+2i}\frac{1-2i}{1-2i} = \frac{5}{1^2+2^2}(1-2i) = 1-2i\\ u &= \text{Re } (1-2i)(\cos t + i\sin t) = \cos t + 2\sin t.\end{aligned}$$

Substituting v and applying the Shift Lemma leads to

$$\begin{aligned}(D^2+2D+2)\cdot e^{-t}(Bt+C) &= \left((D+1)^2+1\right)\cdot e^{-t}(Bt+C)\\ &= e^{-t}\left((D-1+1)^2+1\right)\cdot(Bt+C)\\ e^{-t}(D^2+1)(Bt+C) &= e^{-t}(Bt+C) = e^{-t}(6t+7).\\ v &= e^{-t}(6t+7).\end{aligned}$$

3. The state variable y of a system is governed by the differential equation $y'' + y = f(t)$, in which f is the "switch" function

$$f(t) = \begin{cases} 0 & 0 \le t < a \\ 1 & a \le t < b \\ 0 & b \le t \end{cases}$$

that is "on" for $a \le t < b$.

(a) Express f in terms of the Heaviside function H.

Solution: A unit step "up" at $t = a$ followed by a unit step "down" at $t = b$ is $f(t) = H(t-a) - H(t-b)$.

(b) Use the Laplace transform method to find the state $y(t)$ at any time t, given that the system is at rest at $t = 0$: $y(0) = y'(0) = 0$.

Solution: Apply the Laplace transform to the equation.

$$\begin{aligned}s^2Y(s) - sy(0) - y'(0) + Y(s) &= \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}\\ Y(s) &= (e^{-as} - e^{-bs})\frac{1}{s(s^2+1)}.\end{aligned}$$

By partial fractions

$$\begin{aligned} \frac{1}{s(s^2+1)} &= \frac{A}{s} + \frac{Bs+C}{s^2+1} \\ 1 &= As^2 + A + Bs^2 + Cs \end{aligned}$$

so $A = 1$, $B = -1$, $C = 0$ and $Y(s) = (e^{-as} - e^{-bs})\left(\frac{1}{s} - \frac{s}{s^2+1}\right)$.
Taking the inverse Laplace transform,

$$y = H(t-a) - H(t-b) - (H(t-a)\cos(t-a) - H(t-b)\cos(t-b)) .$$

(c) What relation between the switching times a and b is necessary for the system to return to rest for $t > b$? [The trigonometric identity $\cos x - \cos y = -2\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)$ is useful.]

Solution: The solution is identically zero for $t > b$ provided

$$\cos(t-a) - cos(t-b) = 0,$$

or, using the supplied identity,

$$-2\sin\frac{1}{2}(2t-a-b)\sin\frac{1}{2}(b-a) = 0.$$

For this to be true for all $t > b$, we must have $\frac{1}{2}(b-a) = n\pi$ with some positive integer n, or $b = a + 2n\pi$.