Math 267 §B2 Midterm Exam Sample Solutions

Carry out the solution of each problem: show steps of any required calculations; state reasons that justify any conclusions. Mere oracular answers will receive no credit.

1. Consider the equation for the velocity of a falling body with air resistance,

\[ mv' = -mg - rv \]

with positive constants \( m \) (mass), \( g \) (acceleration of gravity) and \( r \) (coefficient of friction in air).

(a) Find the equilibrium solution.

Solution: Since this equation is autonomous the equilibrium solution is the constant solution of \(-mg - rv = 0\), that is, \( v = -mg/r \).

(b) Is the equilibrium stable?

Solution: Since \( \frac{d}{dv}(-mg - rv) = -r < 0 \), the equilibrium is stable.

(c) What is the physical meaning of the equilibrium?

Solution: This equilibrium is the terminal velocity. It is the velocity at which the gravity force \(-mg\) is balanced by the air resistance force \(-rv\).

2. Find a particular solution of the differential equation

\[ y'' + 2y' + 2y = 5 \cos t + e^{-t}(6t + 7) \]

by the method of undetermined coefficients.

Solution: By superposition, a particular solution may be written as \( y_p = u + v \), with

\[ u'' + 2u' + 2u = 5 \cos t, \quad v'' + 2v' + 2v = e^{-t}(6t + 7). \]

The characteristic roots satisfy \( r^2 + 2r + 2 = 0 \) or \((r + 1)^2 = -1\), so they are \( r = -1 \pm i \). Neither \( i \) nor \(-1\) is a characteristic root, so the particular solutions have the form \( u = Re^{it}, \ v = e^{-t}(Bt + C) \).
Substituting the form of $u$ into the differential equation gives

$$(D^2 + 2D + 2) \cdot Ae^{it} = (i^2 + 2i + 2)Ae^{it} = 5e^{it}$$

$$(1 + 2i)A = 5\Rightarrow A = \frac{5}{1 + 2i} \frac{1 - 2i}{1 - 2i} = \frac{5}{1^2 + 2^2} (1 - 2i) = 1 - 2i$$

$$u = \Re (1 - 2i)(\cos t + i \sin t) = \cos t + 2 \sin t.$$  

Substituting $v$ and applying the Shift Lemma leads to

$$(D^2 + 2D + 2) \cdot e^{-t}(Bt + C) = \left( (D + 1)^2 + 1 \right) \cdot e^{-t}(Bt + C)$$

$$= e^{-t} \left( (D - 1 + 1)^2 + 1 \right) \cdot (Bt + C)$$

$$e^{-t}(D^2 + 1)(Bt + C) = e^{-t}(Bt + C) = e^{-t}(6t + 7).$$

$$v = e^{-t}(6t + 7).$$

3. The state variable $y$ of a system is governed by the differential equation $y'' + y = f(t)$, in which $f$ is the “switch” function

$$f(t) = \begin{cases} 
0 & 0 \leq t < a \\
1 & a \leq t < b \\
0 & b \leq t 
\end{cases}$$

that is “on” for $a \leq t < b$.

(a) Express $f$ in terms of the Heaviside function $H$.

Solution: A unit step “up” at $t = a$ followed by a unit step “down” at $t = b$ is $f(t) = H(t-a) - H(t-b)$.

(b) Use the Laplace transform method to find the state $y(t)$ at any time $t$, given that the system is at rest at $t = 0$: $y(0) = y'(0) = 0$.

Solution: Apply the Laplace transform to the equation.

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \frac{e^{-as}}{s} - \frac{e^{-bs}}{s}$$

$$Y(s) = \frac{(e^{-as} - e^{-bs})}{s(s^2 + 1)}.$$
By partial fractions

\[
\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}
\]

so \( A = 1, B = -1, C = 0 \) and \( Y(s) = (e^{-as} - e^{-bs}) \left( \frac{1}{s} - \frac{s}{s^2 + 1} \right) \).

Taking the inverse Laplace transform,

\[
y = H(t - a) - H(t - b) - (H(t - a) \cos(t - a) - H(t - b) \cos(t - b))
\]

(c) What relation between the switching times \( a \) and \( b \) is necessary for the system to return to rest for \( t > b \)? [The trigonometric identity

\[
\cos x - \cos y = -2 \sin \frac{1}{2}(x + y) \sin \frac{1}{2}(x - y)
\]

is useful.]

Solution: The solution is identically zero for \( t > b \) provided

\[
\cos(t - a) - \cos(t - b) = 0,
\]

or, using the supplied identity,

\[
-2 \sin \frac{1}{2}(2t - a - b) \sin \frac{1}{2}(b - a) = 0.
\]

For this to be true for all \( t > b \), we must have \( \frac{1}{2}(b - a) = n\pi \) with some positive integer \( n \), or \( b = a + 2n\pi \).