A little over 1000 years ago in Baghdad, a court mathematician named Ibn Sahl investigated and solved the following problem:

Suppose a burning lens presents a flat face to the sun; what must be the shape of the other face that will focus all the rays at a single point?

Your task in this project is to re-derive his solution using two tools developed by European science 700 years after Ibn Sahl:

(i) Snel’s Law of refraction relates the angles made by incident and refracted light rays at an interface between two media.

(ii) Differential and Integral Calculus gives a general method to determine a curve from known properties of its tangent.

The lens is positioned as shown in the crude picture below. Its flat side, perpendicular to the $x$-axis, faces right. On the left the convex face of the lens is a surface of revolution generated by revolving a segment of a curve $y = f(x)$ about the $x$-axis. The sun’s rays enter the lens from the right, parallel to the $x$-axis, and are refracted to the point $F(0,0)$ at the coordinate origin.

1. [The solution to this problem is to be handed in Thursday, January 27; the results are needed by the solvers of parts 2, 3, and 4.] Draw a diagram showing a point $P(x_0,y_0)$ on the curve together with the normal line to the curve $y = f(x)$ at $P$ and the incident and refracted rays at $P$. Show the $x$-intercept of the normal line at $P$ as a point $N$ on the $x$-axis. Label $P$, $F$, $N$ and the angles of incidence and refraction.
Determine the Cartesian coordinates of $N$ in terms of $x_0$, $y_0$ and the slope $m_0 = f'(x_0)$. Express the lengths of the segments $PF$ and $NF$ also in terms of $x_0$, $y_0$ and $m_0$.

2. Let $n$ denote the index of refraction between air and glass, so that $n > 1$. Since light leaving the lens at $P$ travels from glass to air, apply Snell’s Law in the form

$$\sin i = \frac{1}{n} \sin r. \quad (1)$$

Show that (1) implies

$$NF = n \cdot PF. \quad (2)$$

[One way to do this is to use the Law of Sines in the triangle $FPN$, together with the fact that the incident ray is parallel to the $x$-axis.]

3. Put the expressions derived in Problem 1 for $NF$ and $PF$ into (2), and write $x$, $y$, $y'$ for $x_0$, $y_0$ and $m_0$, respectively. The result is a differential equation for $y$. Find the solution of this differential equation in implicit form in terms of $x$, $y$, the refractive index $n$ and an integration constant $C$. [Substitute $y = xv$ to obtain a separable equation for $v$.]

4. Show that the solution derived in Problem 3 represents a family of hyperbolas, by putting it into the form

$$\frac{(x-f)^2}{a^2} - \frac{y^2}{b^2} = 1, \quad f^2 = a^2 + b^2. \quad (3)$$

To do this you must express $a$ and $b$ in terms of the refractive index $n$ and the integration constant $C$. Then answer the following.

(a) What is the significance of the parameter $f$ in Equation (3)?

(b) The hyperbolas (3) are not in standard position. Where are their vertices and foci?

(c) Ibn Sahl announced his solution in the form

$$\frac{f}{a} = n.$$

What does the quantity $f/a$ represent in the geometry of the hyperbola? Verify that Ibn Sahl’s equation is valid for your solution.