Math 267 Final Exam

Carry out the solution of each problem: show steps of any required calculations; state reasons that justify any conclusions. Mere oracular answers will receive no credit.

1. Three 200-liter tanks full of salt solution are connected in a ring. Tank 1 contains 10 kg of salt; Tank 2 and Tank 3 each contain 1 kg of salt. Well-mixed salt solution is pumped from Tank 1 to Tank 2, from Tank 2 to Tank 3, and from Tank 3 to Tank 1 at a rate of 4 liters per minute.

   (a) Formulate a system of differential equations \( \mathbf{q}' = \mathbf{Aq} \) for the amounts of salt in the three tanks.

   (b) Find the equilibrium distribution of salt.

   (c) The matrix \( \mathbf{A} \) has an eigenvalue 0. Find the corresponding eigenvector. To what distribution of salt in the tanks does it correspond?

2. For the equation \( y'' - 2xy' + 4y = 0 \),

   (a) Find the recursion relation for the coefficients in a solution expressed as a power series in \( x \).

   (b) Find a solution that is a quadratic polynomial in \( x \).

3. A 2 kg mass is suspended vertically on a spring. A second 1 kg mass is suspended on a spring attached to the first mass. Let \( x_1 \) and \( x_2 \) be the displacements of the two masses, measured vertically downward from their equilibrium positions. Derive the differential equations for the motion of the two masses if the spring constants are both 4 N/m.

4. Find the solution of the differential equation \( y'' + y = g(t) \) with initial conditions \( y(0) = y'(0) = 0 \), if \( g(t) \) is given by

   \[
g(t) = \begin{cases} 
   \sin t, & 0 < t \leq 2\pi; \\
   0, & 2\pi < t.
   \end{cases}
   \]

5. The system \( x' = \mathbf{Ax} \), with

   \[
   \mathbf{A} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}
   \]

has a critical point at the origin. Determine whether this critical point is asymptotically stable, stable, or unstable, and classify it as to type. Compute \( \exp(\mathbf{At}) \).