Math 201 2nd Midterm Exam **Solutions**

1. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the identity function defined by $f(x) = x$ for all $x \in \mathbb{R}$. Prove directly from the definition of limit that, for any $a \in \mathbb{R}$ we have $\lim_{x\to a} f(x) = a$.

   Let $\varepsilon > 0$ be given. We must find $\delta > 0$ so that $0 < |x - a| < \delta$ implies

   $$|f(x) - a| = |x - a| < \varepsilon.$$

   We choose $\delta = \varepsilon$. Then obviously $|x - a| < \delta$ implies $|x - a| < \varepsilon$ and the proof is complete.

2. (a) (5 points) Complete the definition: "Let $E \subseteq \mathbb{R}$. Then $b \in \mathbb{R}$ is an **upper bound** for E if ..."
   for every $x \in E$, $x \le b$.

   (b) (10 points) Prove: If $A, B \subseteq \mathbb{R}$ and b is an upper bound for A and b is an upper bound for B, then b is an upper bound for $A \cup B$.
   Let $x \in A \cup B$. We must show $x \le b$.
   Since $x \in A \cup B$, then $x \in A$ or $x \in B$.
   If $x \in A$ then $x \le b$ because b is an upper bound for A.
   If $x \in B$ then $x \le b$ because b is an upper bound for B.
   This shows that for any $x \in A \cup B$ we have $x \le b$, so b is an upper bound for $A \cup B$.

3. (20 points) Use the Intermediate Value Theorem to give another proof that every positive real number has a positive real square root. [HINT: Let $a \in \mathbb{R}$ with $a > 0$. Let $g(x) = x^2 - a$ and consider $g(0)$ and $g(a + 1)$.]

   We verify that

   (a) g is continuous,

   (b) $g(0) < 0$ and

   (c) $g(a + 1) > 0$.

   It then follows by the intermediate value theorem that there exists $c \in (0, a+1)$ such that $g(c) = c^2 - a = 0$. This c is a positive square root of a.

   g is continuous: Let $x_0 \in \mathbb{R}$. Then

   $$\begin{aligned}
   \lim_{x\to x_0} g(x) &= \lim_{x\to x_0} x^2 - a \\
   &= \lim_{x\to x_0} x^2 - \lim_{x\to x_0} a && \text{(limit of a sum)} \\
   &= \lim_{x\to x_0} x \lim_{x\to x_0} x - \lim_{x\to x_0} a && \text{(limit of a product)} \\
   &= x_0 \cdot x_0 - a && \text{(Problem 1; limit of a constant)} \\
   &= g(x_0).
   \end{aligned}$$

   We have $g(0) < 0$: Compute $g(0) = 0^2 - a = -a < 0$ because $a > 0$.

   Lastly, $g(a+1) = (a+1)^2 - a = a^2 + a + 1 > 0$ because $1 > 0$ (true in ordered fields), $a > 0$ (by assumption) and $a^2 > 0$ (nonzero square in an ordered field), and the sum of positive numbers is positive.

4. (a) (5 points) Complete the definition: "A function $f : A \to B$ is an **injection** if ..."
   for every $x_1, x_2 \in A$, $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$.

   (b) (5 points) Complete the definition: "A function $f : \mathbb{R} \to \mathbb{R}$ is **increasing** if ..."
   for every $x_1, x_2 \in \mathbb{R}$, $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

   (c) (15 points) Prove: If $f : \mathbb{R} \to \mathbb{R}$ is increasing, then f is an injection.
   Let $f : \mathbb{R} \to \mathbb{R}$ be increasing, and let $x_1, x_2 \in \mathbb{R}$ such that $x_1 \neq x_2$. Then either $x_1 < x_2$ or $x_2 < x_1$.
   If $x_1 < x_2$ then $f(x_1) < f(x_2)$ because f is increasing, so $f(x_1) \neq f(x_2)$.
   If $x_2 < x_1$ then $f(x_2) < f(x_1)$ because f is increasing, so again $f(x_1) \neq f(x_2)$.
   Thus $x_1 \neq x_2$ implies $f(x_1) \neq f(x_2)$ in every case, so f is an injection.

5. (a) (5 points) Complete the definition: "A function $f : I \to \mathbb{R}$ (with $I \subseteq \mathbb{R}$ an open interval) has a **local maximum** at $a \in I$ if ..."
   there exists a radius $r > 0$ such that $0 < |x - a| < r$ implies $f(x) < f(a)$.
   [It is also acceptable to write "$f(x) \leq f(a)$".]

   (b) (15 points) State a theorem that gives the first derivative test for a local maximum [It is *not* required that you prove the theorem].
   Theorem 1 *Let $f : I \to \mathbb{R}$ with $I \subseteq \mathbb{R}$ an open interval, and let $a \in I$. Then f has a local maximum at a if $f'(a) = 0$ and there exists a radius $r > 0$ such that*

   (i) for every $x \in (a - r, a)$, $f'(x) > 0$, and

   (ii) for every $x \in (a, a + r)$, $f'(x) < 0$.