# Math 201 1st Midterm Exam — Solutions

**For full credit show the complete solution of each problem including steps of calculations. Support answers by citing known facts, definitions and theorems. No credit for mere answers unsupported by calculations or reasons.**

1. Consider statements $P, Q, R, S$.
   (a) (5 points) Find a disjunction logically equivalent to $S \to R$.
   $R \vee \neg S$, since the conditional statement is true unless $S$ is true and $R$ is false.
   (b) (5 points) Find a conjunction logically equivalent to $\neg(P \vee Q)$.
   $\neg P \wedge \neg Q$ (De Morgan's Law).
   (c) (5 points) Find a conditional statement logically equivalent to $R \vee (\neg P)$.
   $P \to R$ (or its contrapositive, $\neg R \to \neg P$!) for the same reason as in part (a).
   (d) (10 points) Show *without using truth tables* that $(P \vee Q) \to R$ is logically equivalent to $(P \to R) \wedge (Q \to R)$. [HINT: Use (a), (b), a distributive law, and (c).]

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\begin{aligned}
(P \vee Q) \to R &\equiv R \vee \neg(P \vee Q) && \text{by (a)} \\
&\equiv R \vee (\neg P \wedge \neg Q) && \text{by (b)} \\
&\equiv (R \vee \neg P) \wedge (R \vee \neg Q) && \text{distributive law} \\
&\equiv (P \to R) \wedge (Q \to R) && \text{by (c),twice.}
\end{aligned}
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2. Complete the definition: "A field $k$ is an **ordered field** if there is an order relation '$<$' defined on $k$ that has the following properties: ..."
   (a) (5 points) [**Trichotomy**] For all $a, b \in k$, exactly one of $a < b$, $a = b$, $b < a$ is true.
   (b) (5 points) [**Transitivity**] For all $a, b, c \in k$, if $a < b$ and $b < c$ then $a < c$.
   (c) (5 points) [**Addition**] For all $a, b, c \in k$, if $a < b$ then $a + c < b + c$.
   (d) (5 points) [**Multiplication**] For all $a, b, c \in k$, if $a < b$ and $0 < c$ then $ac < bc$.

3. (10 points) Recall that an element $a$ of an ordered field is **positive** if $0 < a$. Prove directly from the definition of ordered field: an element $a$ in an ordered field is positive if and only if $-a < 0$.

   *Proof:* Let $a$ be an element of an ordered field.

   To prove necessity, assume $0 < a$. Using the addition property (c), add $-a$ to this inequality and simplify to find $-a < a + (-a) = 0$.

   To prove sufficiency, assume $-a < 0$. By the addition property (c) again, adding $a$ to this inequality and simplifying gives $0 < a$.

4. (a) (5 points) Complete the definition: "Let $m$ and $n$ be integers with $n \neq 0$. We say that $n$ **divides** $m$ if ...."

   there exists an integer $q$ such that $m = qn$.

   (b) (5 points) Complete the definition: "Let $a$ and $b$ be integers, and $n$ be a natural number. We say that $a$ is **congruent** to $b$ (mod $n$) if ...."

   $n$ divides $b - a$.

   (c) (15 points) Prove: if $a$ and $b$ are consecutive even integers, then 4 divides one of them and does not divide the other.

   *Proof:* Since $a$ and $b$ are consecutive even integers, there exists an integer $q$ such that $a = 2q$, and then $b = 2q + 2$.

   Since $q$ must be either even or odd, consider the two cases separately.

   If $q$ is even there exists an integer $r$ such that $q = 2r$. Then $a = 2q = 4r$ is divisible by 4. Also, $b = a + 2 = 4r + 2$ is congruent to 2 (mod 4), so $b$ is *not* divisible by 4.

   If on the other hand $q$ is odd there exists an integer $s$ such that $q = 2s + 1$. Then $a = 2q = 4s + 2$ is congruent to 2 (mod 4), so $a$ is *not* divisible by 4. Also $b = a + 2 = 4s + 4 = 4(s + 1)$ *is* divisible by 4.

5. (20 points) Prove: If $A$ and $B$ are subsets of a universal set $U$, then

$$A - B \subseteq (A \cup B) - (A \cap B).$$

*Proof:* Let $x \in A - B$. We will prove $x \in (A \cup B) - (A \cap B)$.

Since $x \in A - B$, $x$ belongs to $A$ and $x$ does not belong to $B$.

Because $x \in A$ and $A \subseteq A \cup B$, it follows that $x \in A \cup B$.

On the other hand, $x \notin A \cap B$ since $x \notin B$.

We have shown that $x \in A \cup B$ and $x \notin A \cap B$. Therefore $x \in (A \cup B) - (A \cap B)$.