

MATH 201 §A: BINOMIAL COEFFICIENTS
DUE FRIDAY 27 MAR 2009

Definition 1. For every integer $n \geq 0$ and every integer k , the binomial coefficient $\binom{n}{k}$ is the coefficient of x^k in the expansion of the binomial power $(1+x)^n$.

The binomial expansion is, according to this definition,

$$(1) \quad (1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k.$$

Note that since $(1+x)^0 = 1$ we have $\binom{0}{0} = 1$. We also have

$$(2) \quad \binom{n}{k} = 0 \quad \text{whenever } k < 0 \text{ or } k > n.$$

Our first theorem says that a table of binomial coefficients can be computed one row at a time by the ‘‘Pascal triangle’’ rule.

Theorem 1. For every $n \geq 1$ and every k , the binomial coefficients satisfy

$$(3) \quad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof. Let $n \geq 1$ and write

$$(4) \quad (1+x)^n = (1+x) \cdot (1+x)^{n-1} = (1+x)(1 + \cdots + x^{n-1}).$$

The coefficient of x^0 is 1 on both sides of this equation, so

$$\binom{n}{0} = \binom{n-1}{0} = \binom{n-1}{-1} + \binom{n-1}{0}$$

by Equation (2); this proves Equation (3) in the case $k = 0$. The coefficient of x^n is 1 also, proving the formula in the case $k = n$.

For the coefficient of x^k with $0 < k < n$, rewrite Equation (4):

$$(5) \quad (1+x)^n = (1+x) \left(1 + \cdots + \binom{n-1}{k-1} x^{k-1} + \binom{n-1}{k} x^k + \cdots + x^{n-1} \right).$$

The coefficient of x^k on the left side of Equation (5) is $\binom{n}{k}$. On the right side we find two ways to produce an x^k term: either the 1 from

$(1 + x)$ multiplies the term $\binom{n-1}{k}x^k$, or the x from $(1 + x)$ multiplies the term $\binom{n-1}{k-1}x^{k-1}$. Thus the coefficient of x^k on the right hand side is $\binom{n-1}{k-1} + \binom{n-1}{k}$. \square

Binomial coefficients can be used to enumerate subsets of finite sets.

Theorem 2. *The binomial coefficient $\binom{n}{k}$ is the number of k -element subsets of an n -element set.*

Proof. In-class Exercise. Use induction on n . \square

Consult the proof of Theorem 6.15 in the textbook. You may also want to use the following Lemma.

Lemma 1. *Let A be an n -element set and $B = A \cup \{x\}$ an $n + 1$ -element set. If S is a k -element subset of B , then either $S \subseteq A$, or $x \in S$ (so $S - \{x\}$ has $k - 1$ elements) and $S - \{x\} \subseteq A$.*

Proof. Since B consists of A together with one additional element x , every subset of B that does not contain x is a subset of A . Moreover, if a subset of B does contain x , then all the other elements of that subset belong to A . \square

Theorem 2 provides an alternate proof of the textbook's Theorem 6.15.

Corollary. *The number of subsets of an n -element set is 2^n .*

Proof. By Theorem 2 and the binomial expansion Equation (1), the number of subsets of an n -element set is

$$\sum_{k=0}^n \binom{n}{k} = (1 + 1)^n = 2^n.$$

\square

The expression $\binom{n}{k}$ is pronounced “binomial n , k .” Because of Theorem 2 the symbol is often read as “ n choose k .”

Homework assignment: Theorem 3 gives the formula that expresses binomial coefficients in terms of factorials. Use induction to prove Theorem 3. You will need the convention that $0! = 1$.

Theorem 3. *The binomial coefficients are given by*

$$(6) \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

For the proof, use Theorem 1 to show that the set

$$S = \{n \in \mathbf{N} \cup \{0\} : \text{Equation (6) holds for } 0 \leq k \leq n\}$$

is inductive.