

MATH 201 §C: BINOMIAL COEFFICIENTS

1. WHAT THEY ARE

Definition. For every integer $n \geq 0$ and every integer k , the binomial coefficient $\binom{n}{k}$ is the coefficient of x^k in the expansion of the binomial power $(1+x)^n$.

The expression $\binom{n}{k}$ is pronounced “binomial n, k .”

Expanding $(1+x)^n$ produces no exponent less than zero or greater than n , so

$$(1) \quad \binom{n}{k} = 0 \quad \text{unless } 0 \leq k \leq n.$$

Moreover, since $(1+x)^0 = 1$ we have $\binom{0}{0} = 1$.

The binomial expansion is written

$$(2) \quad (1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k.$$

2. THE PASCAL TRIANGLE RELATION

A table of binomial coefficients can be computed row by row using the “Pascal triangle” rule.

Theorem 1. For every $n \geq 1$ and every k , the binomial coefficients satisfy

$$(3) \quad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Proof. We present a direct proof using “sigma gymnastics”. Let $n \geq 1$ and write

$$(4) \quad (1+x)^n = (1+x) \cdot (1+x)^{n-1} = (1+x) \sum_{k=0}^{n-1} \binom{n-1}{k} x^k.$$

Expanding the right-hand side leads to

$$\begin{aligned}(1+x) \sum_{k=0}^{n-1} \binom{n-1}{k} x^k &= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k \\ &= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + \sum_{k=0}^{n-1} \binom{n-1}{k} x^{k+1}.\end{aligned}$$

We shift the index in the last sum: first set $j = k + 1$,

$$(1+x)^n = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + \sum_{j=1}^n \binom{n-1}{j-1} x^j$$

then replace j with k , to get

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + \sum_{k=1}^n \binom{n-1}{k-1} x^k$$

and finally combine the sums,

$$= \binom{n-1}{0} x^0 + \sum_{k=1}^{n-1} \left(\binom{n-1}{k-1} + \binom{n-1}{k} \right) x^k + \binom{n-1}{n-1} x^n.$$

This proves the Pascal triangle formula (3) for $1 \leq k \leq n-1$. For the “boundary” cases $k=0$ and $k=n$ we have

$$\binom{n}{0} = \binom{n-1}{-1} + \binom{n-1}{0}$$

and

$$\binom{n}{n} = \binom{n-1}{n-1} + \binom{n-1}{n}$$

because $\binom{n-1}{-1} = \binom{n-1}{n} = 0$. □

3. COUNTING WITH BINOMIAL COEFFICIENTS

Binomial coefficients can be used to enumerate subsets of finite sets.

Theorem 2. *The binomial coefficient $\binom{n}{k}$ is the number of k -element subsets of an n -element set.*

Proof. In-class Exercise. Prove by induction on n . □

Theorem 2 says that the binomial coefficient $\binom{n}{k}$ is the number of ways to choose k elements from an n -element set. For this reason the expression $\binom{n}{k}$ is often read “ n choose k .”

To prove Theorem 2, use the following Lemma together with the Pascal triangle relation, Theorem 1.

Lemma 1. *Let A be an n -element set and $B = A \cup \{x\}$ an $n+1$ -element set. If S is a k -element subset of B , then S is either a k -element subset of A or the union of $\{x\}$ and a $k - 1$ -element subset of A .*

Proof. Since B consists of A together with one additional element x , every k -element subset of B that does not contain x is a k -element subset of A . On the other hand, if a k -element subset of B contains x , then the other $k - 1$ elements of that subset form a $k - 1$ -element subset of A . \square

Theorem 2 provides an alternate proof of the textbook’s Exercise 3.11.

Corollary. *The number of subsets of an n -element set is 2^n .*

Proof. By Theorem 2 and the binomial expansion Equation (2), the number of subsets of an n -element set is

$$\sum_{k=0}^n \binom{n}{k} = (1 + 1)^n = 2^n.$$

\square

4. THE FACTORIAL FORMULA FOR BINOMIAL COEFFICIENTS

Homework assignment: Theorem 3 gives the formula that expresses binomial coefficients in terms of factorials. Use induction on n to prove Theorem 3. You will need the convention that $0! = 1$.

Theorem 3. *The binomial coefficients are given by*

$$(5) \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

For the proof, use Theorem 1 to show that the set

$$S = \{n \in \mathbf{N} \cup \{0\} : \text{Equation (5) holds for } 0 \leq k \leq n\}$$

is inductive. To understand how the inductive step works, it helps to write out an example like $\binom{6}{3} = \binom{5}{3} + \binom{5}{2}$ in terms of factorials.