MATH 201: MONOTONE FUNCTIONS
DUE MONDAY, 26 NOV 2007

Our first general integrability theorem is due to Newton. We say that a function f is *nondecreasing* if $x_1 < x_2$ implies $f(x_1) \leq f(x_2)$. *Nonincreasing* functions are defined similarly. A function is *monotone* if it is nondecreasing, or if it is nonincreasing.

Theorem 132.1. *Monotone functions are integrable.*

Assume that $f : [a, b] \to \mathbb{R}$ is nondecreasing. We call a partition *uniform* if all subintervals $[x_{i-1}, x_i]$ have the same length. Fill in the "Know-Show" table below to find a uniform partition that satisfies the integrability criterion in the Corollary to Theorem 131.1.

With a uniform partition we have, writing $h = (b - a)/n$,

$$\sum_{i=1}^{n} (M_i - m_i) \cdot (x_i - x_{i-1}) = h \sum_{i=1}^{n} (M_i - m_i).$$

To get to line Q_2 in the know-show table we need to prove line Q_3:

$$M_i = f(x_i), \qquad m_i = f(x_{i-1}) \qquad (i = 1, 2, \ldots, n).$$

(It suffices to prove $M_i = f(x_i)$; the proof for m_i is similar.) Prove this using the fact that f is nondecreasing.

In line P_2 you announce your choice of n. You make this choice by looking ahead to the inequality you want in line Q_1 and applying the Archimedean property. [BEWARE: *constant* functions are nondecreasing. If f is constant, then $f(b) = f(a)$, so be careful in line P_2 to avoid writing an expression for n with $f(b) - f(a)$ in the denominator.]

Step	**Know**	**Reason**
P	$f : [a, b] \to \mathbb{R}$ nondecreasing	Hypothesis
P_1	Let $\epsilon > 0$.	Hypothesis
P_2	Choose $n \in \mathbb{N}$ such that (?)	(?)
P_3	Let $h = (b - a)/n$ Let $P = \{x_i = a + ih \mid 0 \leq i \leq n\}$.	Define a partition.
$\vdots$	$\vdots$	
Q_3	$M_i \doteq \sup_{x_{i-1} \leq x \leq x_i} f(x) = f(x_i)$ $m_i \doteq \inf_{x_{i-1} \leq x \leq x_i} f(x) = f(x_{i-1})$	(?)
Q_2	$U(P, f) - L(P, f) = h \sum_{i=1}^{n} f(x_i) - f(x_{i-1})$	(?)
Q_1	$U(P, f) - L(P, f) = \frac{b-a}{n}(f(b) - f(a)) < \epsilon$.	(?), P_2, P_3
Q	f is integrable on $[a, b]$.	Cor, Thm 131.1
Step	**Show**	**Reason**