MATH 201: 1ST & 2ND DERIVATIVE TESTS
DUE FRIDAY 2 NOV 2007

Prove:

Theorem 103.1 (First Derivative Test)**.** *Let f be a differentiable real-valued function on an open interval $I \subseteq \mathbb{R}$, and let $a \in I$. If $f'(a) = 0$, and if there exists a radius $r > 0$ such that*

- *$f'(x) > 0$ for all $x \in (a, a + r)$; and*
- *$f'(x) < 0$ for all $x \in (a - r, a)$*

then f has a strict local minimum at a.

Suggestion for the proof: Use a proof by contradiction. Suppose there is a point $z \in (a, a + r)$ with $f(z) \leq f(a)$. Since $f'(z) > 0$, you can use the uphill-at-a-point Lemma to find a $y \in (a, z)$ with $f(y) < f(z)$; then also $f(y) < f(a)$.

Now, f is differentiable at a, so it is continuous at a. Let $\epsilon = f(a) - f(y)$. Choose $\delta > 0$ small enough that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Show that for any $x \in (a, a + \delta)$ we have

$$f(x) > f(a) - \epsilon = f(y).$$

But by Theorem 102.2, f is increasing on $(a, a + r)$. We have $x, y \in (a, a + r)$ and $x < y$, so $f(x) < f(y)$–a contradiction.

Also prove:

Corollary (Second Derivative Test)**.** *Let f be a differentiable real-valued function on an open interval $I \subseteq \mathbb{R}$, and let $a \in I$, and assume that $f''(a)$ exists. If $f'(a) = 0$ and $f''(a) > 0$, then f has a local minimum at a.*

Apply the uphill-at-a-point Lemma to f' at the point a and show that f satisfies the hypotheses of the First Derivative Test.