MATH 201: CHAIN RULE, UPHILL AT A POINT LEMMA DUE FRIDAY 19 OCT 2007

This lesson is about applications of Lagrange's Theorem.

Prove the chain rule for the derivative of a composite function.

Theorem 91.1 (Chain Rule)**.** *If f is differentiable at a, and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a)) \cdot f'(a).$$

For the proof, form the difference quotient

$$\frac{g(f(x)) - g(f(a))}{x - a} \tag{1}$$

and show directly that its limit as $x \to a$ exists and has the required value. To evaluate the limit, apply Lagrange's Theorem (Theorem 82.1) to g. Since g is differentiable at $f(a)$ there is a function w defined in an interval about $f(a)$ such that $\lim_{y \to f(a)} w(y) = w(f(a)) = 0$, and for all y in that interval

$$g(y) = g(f(a)) + (g'(f(a)) + w(y))\,(y - f(a)). \tag{2}$$

Substitute equation (2) with $y = f(x)$ into the difference quotient in formula (1) and simplify and apply known limits and limit rules.

Next, we use Lagrange's Theorem to show that if $f'(a) \neq 0$ then the sign of the derivative determines the sign of the increment $f(x) - f(a)$ when x is close enough to a.

Lemma 91.2 (Uphill at a point Lemma)**.** *Let $f : I \to \mathbb{R}$, with $I \subseteq \mathbb{R}$ an open interval; let $a \in I$, and let f be differentiable at a. If $f'(a) \neq 0$ there exists a radius $r > 0$ such that for all x satisfying $0 < |x - a| < r$,*

- *$f(x) - f(a)$ has the* same *sign as $x - a$ if $f'(a) > 0$, and*
- *$f(x) - f(a)$ has the* opposite *sign from $x - a$ if $f'(a) < 0$.*

Proof. Assume I is an open interval, $f : I \to \mathbb{R}$, $a \in I$, f is differentiable at a and $f'(a) \neq 0$.

By Lagrange's Theorem there exists a function $w : I - \{a\} \to \mathbb{R}$ with $\lim_{x \to a} w(x) = 0$ such that, for all x in $I - \{a\}$,

$$f(x) = f(a) + (f'(a) + w(x))(x - a). \tag{3}$$

Since $\lim_{x\to a} w(x) = 0$ and $f'(a) \neq 0$, there exists a radius $r > 0$ such that $0 < |x - a| < r$ implies

$$|w(x)| < \frac{1}{2}|f'(a)|.$$

Thus if $0 < |x - a| < r$ then $f'(a) + w(x)$ has the same sign as $f'(a)$, for if $f'(a) > 0$,

$$f'(a) + w(x) \geq f'(a) - |w(x)| > f'(a) - \frac{1}{2}f'(a) = \frac{1}{2}f'(a) > 0,$$

and if $f'(a) < 0$,

$$f'(a) + w(x) \leq f'(a) + |w(x)| < f'(a) - \frac{1}{2}f'(a) = \frac{1}{2}f'(a) < 0.$$

We now rewrite equation (3) as

$$f(x) - f(a) = (f'(a) + w(x)) \cdot (x - a)$$

and consider x with $0 < |x - a| < r$. Since $f'(a) + w(x)$ has the same sign as $f'(a)$, the desired conclusion follows from the sign rules for multiplication: $f(x) - f(a)$ has the same sign as $x - a$ if $f'(a) > 0$, and the opposite sign if $f'(a) < 0$. □

Warning! It does *not* follow that f is increasing on any interval containing a. Consider the function

$$f(x) = \begin{cases} x + 2x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

I claim f is differentiable at $x = 0$ with $f'(0) = 1$. For

$$\begin{aligned} \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} &= \lim_{x\to 0} \frac{x + 2x^2 \sin(1/x)}{x} \\ &= \lim_{x\to 0} (1 + 2x \sin(1/x)) = 1 \end{aligned}$$

by Theorem 72.2, because $\lim_{x\to 0} x = 0$ and $|\sin(1/x)| \leq 1$ for all $x \neq 0$.

I claim further that f is not increasing on any interval that contains 0. To prove this it suffices to show (see §2.4 **Exercise 5**) that for every $\epsilon > 0$ there exist p, q such that $0 < p < q < \epsilon$ and $f(p) > f(q)$. Given $\epsilon > 0$ we choose $n \in \mathbf{N}$[1] such that $1/(2n\pi) < \epsilon$ and choose $p = 1/(2n\pi)$. Then $f'(p) < 0$, because for $x \neq 0$ we have

$$f'(x) = 1 + 4x \sin(1/x) - 2\cos(1/x)$$

so $f'(p) = -1$. It now follows from Lemma 91.2 that a radius $r > 0$ exists such that for all q with $p < q < r$ we have $f(q) - f(p) < 0$; any q satisfying $p < q < \min\{p + r, \epsilon\}$ will do.

[1]That such an n exists depends on the *Archimedean property* of the natural numbers, about which we will learn shortly.